Partial Derivative & Chain Rule

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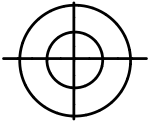
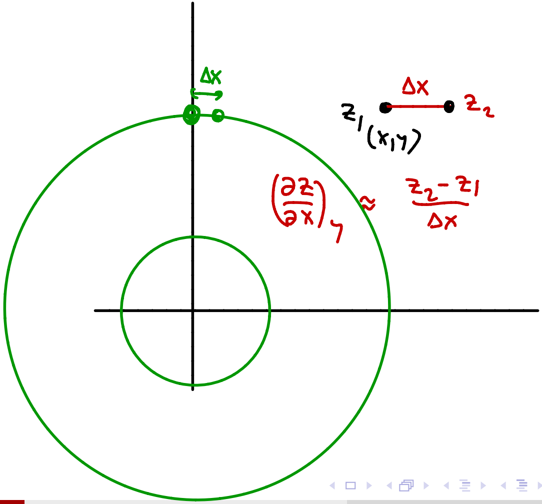
<u>ตัวอย่าง</u> ให้ $z = x^2 + y^2$ จงหา $\left(\frac{\partial z}{\partial x}\right)_y$ และ $\left(\frac{\partial z}{\partial x}\right)_r$ และ $\left(\frac{\partial z}{\partial x}\right)_\theta$

$z = r^2$



$z = x^2 + y^2$

$\left(\frac{\partial z}{\partial x}\right)_y = 2x$



$z_1\ (x,y) \xrightarrow{\Delta x} z_2$

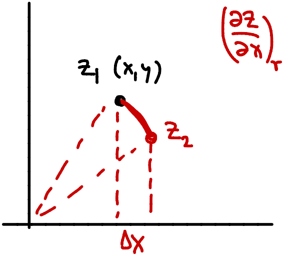
$\left(\frac{\partial z}{\partial x}\right)_y \approx \frac{z_2 - z_1}{\Delta x}$

$$z = x^2 + y^2 \quad [r^2 = x^2 + y^2]$$

$$z = r^2$$

$$z(x, r) = r^2$$

$$\left(\frac{\partial z}{\partial x}\right)_r = 0$$



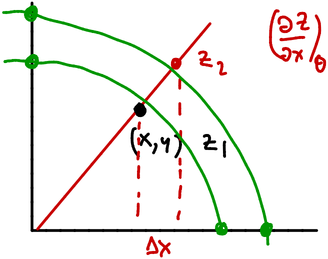
$$\left(\frac{\partial z}{\partial x}\right)_r = \frac{z_2 - z_1}{\Delta x} = 0$$

$$z = x^2 + y^2 \quad \left[\tan\theta = \frac{y}{x}\right]$$

$$= x^2 + x^2 \tan^2\theta$$

$$= \frac{x^2}{\cos^2\theta}$$

$$\left(\frac{\partial z}{\partial x}\right)_\theta = \frac{2x}{\cos^2\theta}$$



$$\left(\frac{\partial z}{\partial x}\right)_\theta = \frac{z_2 - z_1}{\Delta x}$$

<u>ตัวอย่าง</u> ให้ $z = x^2 + y^2$ จงหา $\underbrace{\frac{\partial^2 z}{\partial x \partial r}}$ และ $\underbrace{\frac{\partial^2 z}{\partial x^2}}_{\text{กำกวม}}$

$$\frac{\partial^2 z}{\partial x \partial r} \Downarrow z(x, r)$$

$$\frac{\partial^2 z}{\partial x^2} \text{ (อะไรคงที่)} \Downarrow z(x, ?)$$

$$\cancel{\frac{\partial^2 z}{\partial x \partial r}} = \left(\frac{\partial}{\partial x} \left[\left(\frac{\partial z}{\partial r} \right)_y \right] \right)_\theta$$

Notation

$z(x_1, x_2, \dots)$

$$\left(\frac{\partial z}{\partial x}\right)_y$$

$\Uparrow$

y คงที่

$\partial_x z \;,\; z_x \;,\; z_1 \;\Rightarrow\; \dfrac{\partial z}{\partial x}$

$\partial_y z \;,\; z_y \;,\; z_2 \;\Rightarrow\; \dfrac{\partial z}{\partial y}$

$\partial_{xy} z \;,\; z_{xy} \;,\; z_{12} \;\Rightarrow\; \dfrac{\partial^2 z}{\partial y \partial x}$

$\Rightarrow\; \dfrac{\partial^2 z}{\partial x^2}$

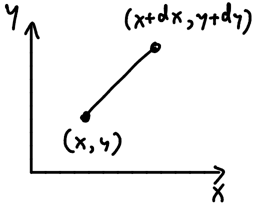
$\Rightarrow\; \dfrac{\partial^2 z}{\partial y^2}$

Total Differential

$$z = z(x,y)$$

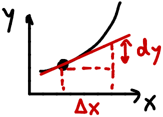
$$\boxed{dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$

$$dz \approx z(x+dx,\,y+dy) - z(x,y)$$

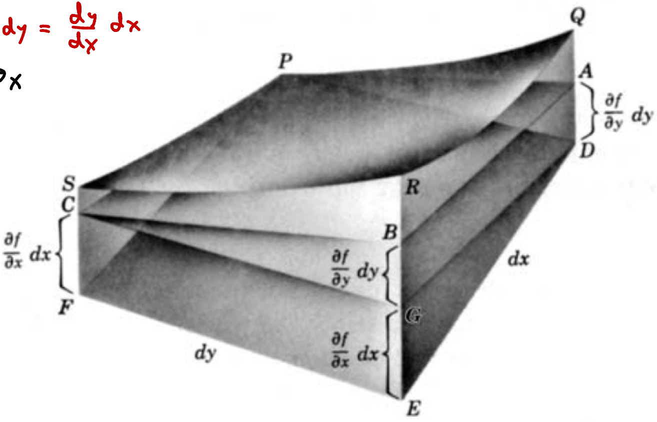


Total Differential

1 ตัวแปร $y = f(x)$

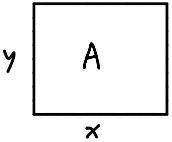


$dy = \frac{dy}{dx}\,dx$



$ER = \Delta z$
$EB = dz$

ตัวอย่าง คำนวณพื้นที่สี่เหลี่ยมผืนผ้าจาก $A = xy$ จงหาความคลาดเคลื่อนของ A ในรูปของความคลาดเคลื่อนของ x และ y



$$\Delta A = \frac{\partial A}{\partial x}\Delta x + \frac{\partial A}{\partial y}\Delta y$$

$$\Delta A = y\,\Delta x + x\,\Delta y$$

$$\frac{\Delta A}{A} = \frac{\Delta x}{x} + \frac{\Delta y}{y}$$

[สูตรที่ถูก

$$\frac{\Delta A}{A} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$
]

<u>ตัวอย่าง</u> คำนวณคาบของเพนดูลัมได้จาก $T = 2\pi\sqrt{\frac{l}{g}}$ จงหาความคลาดเคลื่อนของ g ในรูปความคลาดเคลื่อนของ l และ T

$$g = 4\pi^2 \frac{\ell}{T^2}$$

$$\ln g = \ln(4\pi^2) + \ln \ell - 2\ln T$$

$$\frac{\Delta g}{g} = \frac{\Delta \ell}{\ell} - 2\frac{\Delta T}{T}$$

$$\Downarrow$$

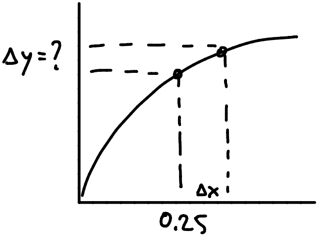
$$\frac{\Delta g}{g} = \left|\frac{\Delta \ell}{\ell}\right| + 2\left|\frac{\Delta T}{T}\right|$$

ตัวอย่าง จงหา $\sqrt{0.25 + 10^{-20}} - \sqrt{0.25}$

$$\Downarrow \qquad \Downarrow$$

0.500 1   0.5

$$\begin{array}{r} 0.5000\ldots1 \\ - \quad 0.5000\ldots0 \end{array}$$



$$f(x) = \sqrt{x}$$

$$df = \frac{df}{dx}dx = \frac{1}{2\sqrt{x}}dx$$

$$= \frac{1}{2\sqrt{0.25}} 10^{-20} = 10^{-20}$$

$$\sqrt{0.25+\epsilon} = \sqrt{0.25\left(1+\frac{\epsilon}{0.25}\right)}$$

$$= 0.5\sqrt{1+4\epsilon}$$

$$\approx 0.5\left(1+\frac{1}{2}(4\epsilon)\right)$$

$$\approx 0.5 + 10^{-20}$$

<u>ตัวอย่าง</u> ใน polar coordinate จงหา $\left(\frac{\partial x}{\partial \theta}\right)_y$ และ $\left(\frac{\partial \theta}{\partial x}\right)_y$

$$x = r\cos\theta \quad , \quad r^2 = x^2 + y^2$$

$$y = r\sin\theta \quad , \quad \tan\theta = \frac{y}{x}$$

$$x = r\cos\theta = \left(\frac{y}{\sin\theta}\right)\cos\theta = y\cot\theta$$

$$\left(\frac{\partial x}{\partial \theta}\right)_y = -y\csc^2\theta$$

$$\theta = \arctan\left(\frac{y}{x}\right) \qquad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\left(\frac{\partial \theta}{\partial x}\right)_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

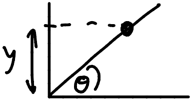
$$= -\frac{y}{x^2+y^2}$$

x, y
r, θ

4 ตัวแปร
2 เงื่อนไข
$\Rightarrow$ 2 ตัวแปรอิสระ

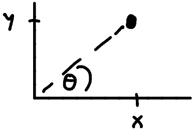
หา $\left(\frac{\partial x}{\partial \theta}\right)_y$

x ตัวแปรตาม
y, θ ตัวแปรต้น



หา $\left(\frac{\partial \theta}{\partial x}\right)_y$

θ ตัวแปรตาม
x, y ตัวแปรต้น



$$\left(\frac{\partial x}{\partial \theta}\right)_y = -y\csc^2\theta = -\frac{y}{\sin^2\theta}$$

$$= -\frac{y}{(y/r)^2}$$

$$= -\frac{r^2}{y}$$

$$\left(\frac{\partial \theta}{\partial x}\right)_y = -\frac{y}{x^2+y^2} = -\frac{y}{r^2}$$

$$\Rightarrow \boxed{\left(\frac{\partial x}{\partial \theta}\right)_y = \frac{1}{\left(\frac{\partial \theta}{\partial x}\right)_y}}$$

[จริง เกือบหมด
ฟังก์ชันทำตัวดี]

[ตัวแปรคงที่ต้องเป็นตัวเดียวกัน]

<u>ตัวอย่าง</u> ให้ $z = x^2 - y^2$ จงหา $\left(\frac{\partial z}{\partial x}\right)_y$ และ $\left(\frac{\partial x}{\partial y}\right)_z$ และ $\left(\frac{\partial y}{\partial z}\right)_x$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$dz = 2x\, dx - 2y\, dy$$

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{dz}{dx}\bigg|_{dy=0} = 2x$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{dx}{dy}\bigg|_{dz=0} = \frac{y}{x}$$

$$\left(\frac{\partial y}{\partial z}\right)_x = \frac{dy}{dz}\bigg|_{dx=0} = -\frac{1}{2y}$$

$$\boxed{\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -1}$$

$$= 2x\left(\frac{y}{x}\right)\left(-\frac{1}{2y}\right)$$

$$= -1$$

<u>ตัวอย่าง</u> (ทบทวน chain rule) จงหา $\frac{dy}{dx}$ เมื่อ $y = \ln(\sin(2x))$

$$\frac{dy}{dx} = \frac{d\ln(\sin(2x))}{d\sin(2x)} \times \frac{d\sin(2x)}{d(2x)} \times \frac{d(2x)}{dx}$$

$$= \frac{1}{\sin(2x)} \times \cos(2x) \times 2 = 2\cot(2x)$$

$$u = \sin v, \quad v = 2x \Rightarrow y = \ln u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

<u>ตัวอย่าง</u> ให้ $z = xy$, $x = \sin(s+t)$, $y = s - t$ จงหา $\left(\frac{\partial z}{\partial s}\right)_t$ และ $\left(\frac{\partial z}{\partial t}\right)_s$

วิธีแรก: เขียน $z(s,t)$

$$z = (s-t)\sin(s+t)$$

$$\left(\frac{\partial z}{\partial s}\right)_t = (s-t)\cos(s+t) + \sin(s+t)$$

$$\left(\frac{\partial z}{\partial t}\right)_s = (s-t)\cos(s+t) - \sin(s+t)$$

[ต้องเป็น explicit function]

z ตัวแปรตาม

$\Uparrow$

xy

$\Uparrow$

st ตัวแปรต้น

ตัวอย่าง

Total differential

$$z = xy \Rightarrow \boxed{dz = y\,dx + x\,dy}$$

$$x = \sin(s+t) \Rightarrow dx = \cos(s+t)(ds + dt)$$

$$\boxed{dx = \cos(s+t)\,ds + \cos(s+t)\,dt}$$

$$y = s - t \rightarrow dy = d(s-t)$$

$$\boxed{dy = ds - dt}$$

$$dz = y\,dx + x\,dy$$

$$dx = \cos(s+t)\,ds + \cos(s+t)\,dt$$

$$dy = ds - dt$$

$$\Rightarrow \quad dz = y\left[\cos(s+t)\,ds + \cos(s+t)\,dt\right] + x\left(ds - dt\right)$$

$$dz = \left[y\cos(s+t) + x\right] ds + \left[y\cos(s+t) - x\right] dt$$

หา $\left(\frac{\partial z}{\partial s}\right)_t$ $\left(\frac{\partial z}{\partial t}\right)_s$

$$\left(\frac{\partial z}{\partial s}\right)_t = \left.\frac{dz}{ds}\right|_{dt=0} = y\cos(s+t) + x$$

$$\left(\frac{\partial z}{\partial t}\right)_s = \left.\frac{dz}{dt}\right|_{ds=0} = y\cos(s+t) - x$$

$$dz = \underbrace{\left[y\cos(s+t)+x\right]}_{\left(\frac{\partial z}{\partial s}\right)_t} ds + \underbrace{\left[y\cos(s+t)-x\right]}_{\left(\frac{\partial z}{\partial t}\right)_s} dt$$

<u>ตัวอย่าง</u> $z = z(x,y)$, $x = x(s,t)$, $y = y(s,t)$ จงหา $\left(\frac{\partial z}{\partial s}\right)_t$ และ $\left(\frac{\partial z}{\partial t}\right)_s$

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$dx = \left(\frac{\partial x}{\partial s}\right)_t ds + \left(\frac{\partial x}{\partial t}\right)_s dt$$

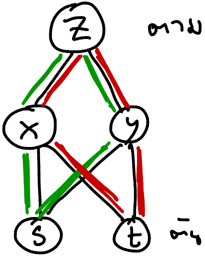
$$dy = \left(\frac{\partial y}{\partial s}\right)_t ds + \left(\frac{\partial y}{\partial t}\right)_s dt$$

$$\Rightarrow \quad dz = \Bigg[\overbrace{\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial s}\right)_t + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial s}\right)_t}^{\left(\frac{\partial z}{\partial s}\right)_t} \Bigg] ds$$

$$+ \Bigg[\underbrace{\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial t}\right)_s + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial t}\right)_s}_{\left(\frac{\partial z}{\partial t}\right)_s} \Bigg] dt$$

$$\left(\frac{\partial z}{\partial s}\right)_t = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial s}\right)_t + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial s}\right)_t$$

$$\left(\frac{\partial z}{\partial t}\right)_s = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial t}\right)_s + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial t}\right)_s$$



เมทริกซ์

$$\begin{bmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

ตัวอย่าง ให้ $z = xy$, $x = \sin(s+t)$, $y = s - t$ จงหา $\left(\frac{\partial s}{\partial z}\right)_x$

$$\begin{array}{c} z \\ \Uparrow \\ xy \\ \uparrow \\ st \end{array} \quad \Rightarrow$$

z x y s t

5 ตัวแปร

3 เงื่อนไข

$\Rightarrow$ 2 ตัวแปรต้น (อิสระ)

$$\Rightarrow \quad \begin{array}{cl} s & \text{ตาม} \\ \Uparrow & \\ yt & \\ \Uparrow & \\ zx & \text{ต้น} \end{array}$$

$$z = xy \quad \Rightarrow \quad dz = y\,dx + x\,dy \qquad -①$$

$$x = \sin(s+t) \quad \Rightarrow \quad dx = \cos(s+t)\,ds + \cos(s+t)\,dt \qquad -②$$

$$y = s - t \quad \Rightarrow \quad dy = ds - dt \qquad -③$$

- 3 สมการ ในรูป dz, dx, dy, ds, dt

- กำจัด 2 ตัวแปร (dy, dt)

หา $\left(\frac{\partial s}{\partial z}\right)_x$

เขียน $s = s(z,x)$

$\Rightarrow ds = \dots dz + \dots dx$

$$\Rightarrow \quad ds = \underbrace{\left[\ \dots\ \right]}_{\left(\frac{\partial s}{\partial z}\right)_x} dz + \underbrace{\left[\ \dots\ \right]}_{\left(\frac{\partial s}{\partial x}\right)_z} dx$$

<u>ตัวอย่าง</u> (ทบทวน implicit differentiation) จงหา $\frac{dy}{dx}$ เมื่อ $y + e^y = x$

<u>ตัวอย่าง</u> ให้ $z+\sin z = x^2+xy,\ x^2+y^3 = st,\ x^3-y^2 = s^2+t^2$ จงหา $\left(\frac{\partial z}{\partial s}\right)_t$ และ $\left(\frac{\partial z}{\partial t}\right)_s$ และ $\boxed{\left(\frac{\partial s}{\partial z}\right)_x}$

$z+\sin z = x^2+xy \Rightarrow dz + \cos z\, dz = 2x\,dx + x\,dy + y\,dx$

$x^2+y^3 = st \Rightarrow 2x\,dx + 3y^2\,dy = s\,dt + t\,ds$

$x^3-y^2 = s^2+t^2 \Rightarrow 3x^2\,dx - 2y\,dy = 2s\,ds + 2t\,dt$

หา $\left(\frac{\partial s}{\partial z}\right)_x$ จัดรูป กำจัด $dy, dt \Rightarrow ds = [\ldots]\,dz + [\ldots]\,dx$

หา $\left(\frac{\partial z}{\partial s}\right)_t \Rightarrow$ กำจัด $dx, dy \Rightarrow dz = [\ldots]\,ds + [\ldots]\,dt$

Quiz

กำหนดให้ $z = xy + \sin x \cos y$ จงหา

1. $\dfrac{\partial z}{\partial x}$

2. $\dfrac{\partial z}{\partial \cancel{x}\, y}$

3. $\dfrac{\partial^2 z}{\partial x^2}$

4. $\dfrac{\partial^2 z}{\partial y^2}$

5. $\dfrac{\partial^2 z}{\partial x \partial y}$