19 - 22 พย 63

Change of variables
Differentiation of Integrals
(Very Basic) Calculus of Variation

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การเปลี่ยนตัวแปรในสมการเชิงอนุพันธ์ย่อย

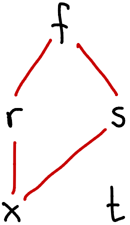
<u>ตัวอย่าง</u> จงเปลี่ยนสมการ $\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial \mathbf{t}^2} = 0$ ให้อยู่ในรูปตัวแปร r, s โดยที่ $r = x - vt$ และ $\mathsf{S} = x + vt$

หา $f(x,t)$ $\Rightarrow$ หา $f(r,s)$

พยายามเขียน $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial t^2}$ ในรูป r,s

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cancelto{1}{\frac{\partial r}{\partial x}} + \frac{\partial f}{\partial s}\cancelto{1}{\frac{\partial s}{\partial x}}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}$$



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)\cancel{\frac{\partial r}{\partial x}}^{=1} + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right)\cancel{\frac{\partial s}{\partial x}}^{=1}$$

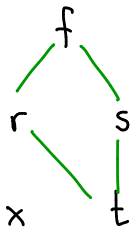
$$= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right) + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)$$

$$= \frac{\partial^2 f}{\partial r^2} + 2\frac{\partial^2 f}{\partial r \partial s} + \frac{\partial^2 f}{\partial s^2}$$

$$\left[\begin{array}{l} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial r} + \dfrac{\partial}{\partial s} \\[2ex] \dfrac{\partial^2 f}{\partial x^2} = \left(\dfrac{\partial}{\partial x}\right)^2 f = \left(\dfrac{\partial}{\partial r} + \dfrac{\partial}{\partial s}\right)\left(\dfrac{\partial}{\partial r} + \dfrac{\partial}{\partial s}\right) f \end{array}\right]$$

Operator

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial r}\underbrace{\frac{\partial r}{\partial t}}_{-v} + \frac{\partial f}{\partial s}\underbrace{\frac{\partial s}{\partial t}}_{+v}$$



$$= -v\frac{\partial f}{\partial r} + v\frac{\partial f}{\partial s}$$

$$= \left(-v\frac{\partial}{\partial r} + v\frac{\partial}{\partial s}\right) f$$

$$\frac{\partial^2 f}{\partial t^2} = \left(-v\frac{\partial}{\partial r} + v\frac{\partial}{\partial s}\right)^2 f$$

$$= v^2\frac{\partial^2 f}{\partial r^2} - 2v^2\frac{\partial^2 f}{\partial r\partial s} + v^2\frac{\partial^2 f}{\partial s^2}$$

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

$$\Downarrow$$

$$\frac{\partial^2 f}{\partial r^2} + 2\frac{\partial^2 f}{\partial r \partial s} + \frac{\partial^2 f}{\partial s^2} - \frac{1}{v^2}\left(v^2\frac{\partial^2 f}{\partial r^2} - 2v^2\frac{\partial^2 f}{\partial r \partial s} + v^2\frac{\partial^2 f}{\partial s^2}\right) = 0$$

$$4\frac{\partial^2 f}{\partial r \partial s} = 0$$

$$f(r,s) = g(r) + h(s)$$

ฟังก์ชันอะไรก็ได้

$$f(x,t) = g(x-vt) + h(x+vt)$$

คลื่นไปทาง $+x$

คลื่นไปทาง $-x$

ตัวอย่าง จงเปลี่ยน Laplace equation $\underbrace{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}}$ $= 0$ ในอยู่ในรูป polar coordinate

$$\Downarrow$$

$$r, \theta$$

$$\Downarrow$$

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0$$

$$f = f(x,y)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$df = p\,dx + q\,dy \quad -①$$

$\left[\ p = \frac{\partial f}{\partial x} \quad q = \frac{\partial f}{\partial y}\right.$

ของ p, q เป็นตัวแปรอิสระ $\left.\right]$

$$d(qy) = q\,dy + y\,dq \quad -②$$

$$d(\underbrace{f - qy}_{\text{ฟังก์ชัน } g}) = p\,dx - y\,dq$$

ฟังก์ชัน $g \Rightarrow g(x,q) \Rightarrow dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial q}dq$

$$\left.\begin{aligned} p &= \left(\frac{\partial g}{\partial x}\right)_q \\ -y &= \left(\frac{\partial g}{\partial q}\right)_x \end{aligned}\right\} \Rightarrow \left(\frac{\partial p}{\partial q}\right)_x = -\left(\frac{\partial y}{\partial x}\right)_q$$

Maxwell's Relation

$$dU = TdS - pdV \quad -① \qquad [\, dQ = TdS \,]$$

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$$

$$d(TS) = TdS + SdT \quad -②$$

$$d(pV) = pdV + Vdp \quad -③$$

$$①+③ : \quad d(\underbrace{U+pV}_{}) = TdS + Vdp$$

$H =$ Enthalpy $= H(S,p)$

$$\left.\begin{aligned} \left(\frac{\partial H}{\partial S}\right)_p &= T \\ \left(\frac{\partial H}{\partial p}\right)_S &= V \end{aligned}\right\} \qquad \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$$

$$dU = TdS - pdV \quad -①$$

$$d(TS) = TdS + SdT \quad -②$$

$$d(pV) = pdV + Vdp \quad -③$$

$$①-② : \quad d(\underbrace{U-TS}_{F}) = -pdV - SdT$$

F = Helmholtz free energy = $F(V,T)$

$$\left.\begin{aligned} \left(\frac{\partial F}{\partial V}\right)_T &= -p \\ \left(\frac{\partial F}{\partial T}\right)_V &= -S \end{aligned}\right\} \quad \left(\frac{\partial p}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T$$

$$dU = TdS - pdV \quad -①$$

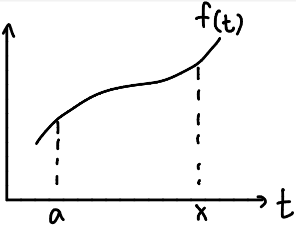
$$d(TS) = TdS + SdT \quad -②$$

$$d(pV) = pdV + Vdp \quad -③$$

① - ② + ③ $$d(\underbrace{U - TS + pV}_{G \,=\, \text{Gibbs free energy} \,=\, G(T,p)}) = -SdT + Vdp$$

Differentiation of Integrals

$$I_1(x) = \int_a^x f(t)\,dt$$



$$I_2(x) = \int_{u(x)}^{v(x)} f(t)\,dt$$

$$\frac{dI_2}{dx} = \,?$$ หายังไง?

$$I(x) = \int_{u(x)}^{v(x)} f(t)\,dt \Rightarrow \frac{dI}{dx} = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$$

<u>ตัวอย่าง</u> จงหา $\frac{dI}{dx}$ เมื่อ $I = \int_0^{x^{1/3}} t^2 \, dt$

ใช้สูตรเมื่อกี้

อินทิเกรต

$\downarrow$

แทนลิมิต

$\downarrow$

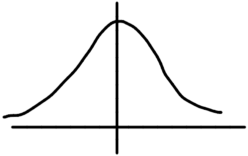
หาอนุพันธ์

$$I(x) = \int_a^b f(x,t)\,dt$$ หา $\frac{dI}{dx}$ ยังไง?

$$\frac{dI}{dx} = \frac{d}{dx}\left[\int_a^b f(x,t)\,dt\right] = \int_a^b \left[\frac{\partial}{\partial x} f(x,t)\right] dt$$

<u>ตัวอย่าง</u> กำหนดให้ $\int_{-\infty}^{\infty} e^{-kx^2}\,dx = \sqrt{\frac{\pi}{k}}$ จงหา $\int_{-\infty}^{\infty} x^2 e^{-kx^2}\,dx$

$$\int_{-\infty}^{\infty} e^{-kx^2}\,dx = \sqrt{\frac{\pi}{k}}$$



$$\frac{d}{dk}\left[\int_{-\infty}^{\infty} e^{-kx^2}\,dx\right] = \frac{d}{dk}\left(\sqrt{\frac{\pi}{k}}\right)$$

$$\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial k} e^{-kx^2}\right)dx = \sqrt{\pi}\left(-\frac{1}{2}k^{-3/2}\right)$$

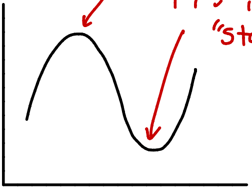
$$\int_{-\infty}^{\infty}\left(-x^2 e^{-kx^2}\right)dx = -\frac{1}{2}\sqrt{\frac{\pi}{k^3}}$$

$$\int_{-\infty}^{\infty} x^2 e^{-kx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{k^3}}$$

Stationary Point ของ Function

$$\frac{dy}{dx} = 0$$

$\Rightarrow$ y สูงสุด/ต่ำสุด/อื่นๆ



Stationary Point ของ Integral

$y' = \frac{dy}{dx}$

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

สำหรับ $y(x)$ ที่ให้ $\Rightarrow$ หาค่า I ได้

เป้าหมายของ Calculus of variation

$\boxed{\text{หา } y(x) \text{ ที่ทำให้ } I \text{ มีค่า มากสุด/น้อยสุด/Stationary}}$

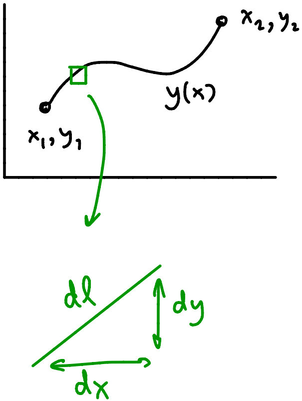
ตัวอย่าง Geodesics

ความยาวของ $y(x)$ ระหว่าง $x_1 \to x_2$

$$l = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$l = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

$y(x) = ?$ ที่ทำให้ l น้อยสุด



$$dl = \sqrt{dx^2+dy^2}$$

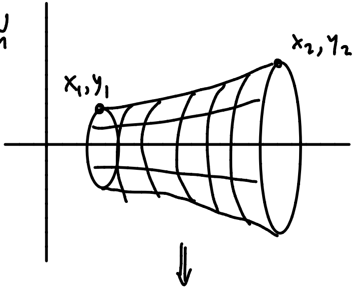
$$dl = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

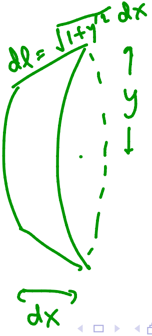
ตัวอย่าง Soap Film problem

รูปร่างเป็นอย่างไร ? => minimize พื้นที่

$$A = \int_{x_1}^{x_2} 2\pi y \sqrt{1+y'^2}\, dx$$

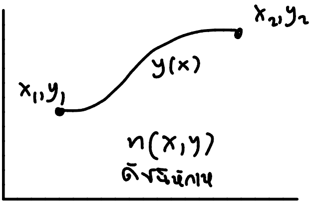
หา $y(x)$ ที่ทำให้ A น้อยสุด





$$dA = 2\pi y\, dl = 2\pi y \sqrt{1+y'^2}\, dx$$

ตัวอย่าง Fermat's Principle



"แสงจะเลือกเส้นทางที่ทำให้ เวลาที่ใช้ stationary"

$$dt = \frac{d\ell}{v(x,y)} = \frac{n(x,y)}{c}\, d\ell$$

$$t = \int_{x_1}^{x_2} \frac{n(x,y)}{c} \sqrt{1+y'^2}\, dx$$

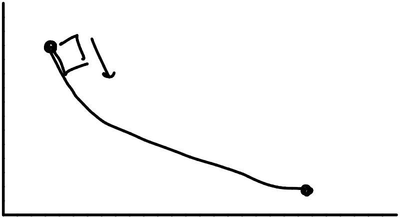
เลือก $y(x)$ ที่ทำให้ t stationary

ตัวอย่าง Brachistochrone problem

ราง รูปร่างยังไง เวลาถึงเร็วสุด

Greek: Brachistos = เร็วสุด

chronos = เวลา



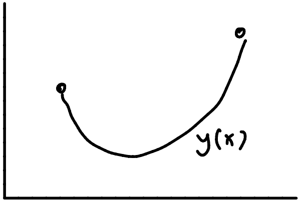
$$t = \int_{x_1}^{x_2} \frac{d\ell}{v} = \int_{x_1}^{x_2} \sqrt{\frac{1+y'^2}{y}}\, dx$$

$v = v(y)$ หาได้จาก อนุรักษ์พลังงาน

$$\frac{1}{2}mv^2 - mgy = 0$$

$$v \propto \sqrt{y}$$

ตัวอย่าง Hanging Rope



หา $y(x)$ ที่ทำให้ PE น้อยสุด
โดยที่ ℓ คงที่

$$PE = \int_{x_1}^{x_2} g y \, dm = \int_{x_1}^{x_2} g y \lambda \, d\ell$$

$$PE = \int_{x_1}^{x_2} g \lambda y \sqrt{1+y'^2} \, dx \Rightarrow \text{น้อยสุด}$$

$$\ell = \int_{x_1}^{x_2} \sqrt{1+y'^2} \, dx \Rightarrow \text{คงที่}$$

Constrained optimization

$\Downarrow$

Lagrange Multiplier

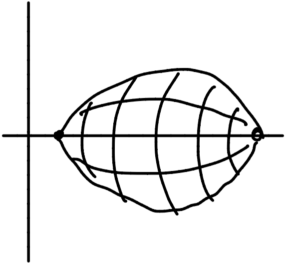
ตัวอย่าง Isoperimetric Problem

คงที่

perimeter = พื้นผิว

พื้นที่ผิวคงที่

ปริมาตรมากสุด



Euler Equation

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

$$\Downarrow$$

Euler Equation $\boxed{\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) - \dfrac{\partial F}{\partial y} = 0}$

$$\Downarrow$$

สมการเชิงอนุพันธ์ (2nd order)

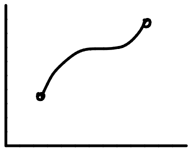
$$\Downarrow$$

แก้ ODE หา $y(x)$

$$\Downarrow$$

Boundary condition (x_1, y_1) & (x_2, y_2)

ตัวอย่าง Geodesics



$$\ell = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

$$F(x,y,y') = \sqrt{1+y'^2}$$

$$\frac{\partial F}{\partial y} = 0 \,, \quad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$\Leftarrow$ 2nd order ODE

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\frac{y'}{\sqrt{1+y'^2}} = C_1$$

$$\Downarrow$$

$$y' = \text{ค่าคงที่} = m$$

$$\frac{dy}{dx} = m$$

$$y = mx + C \quad \Leftarrow \text{เส้นตรง}$$

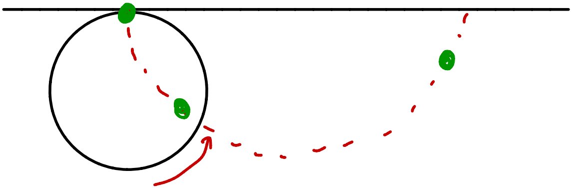
ตัวอย่าง Soap Film Problem

<u>ตัวอย่าง</u> Brachistochrone Problem

$\Rightarrow$ คำตอบคือ cycloid

$$x = a(\theta - \sin\theta)$$

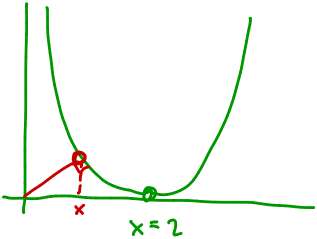
$$y = a(1 - \cos\theta)$$



Quiz

จงหาจุดบนเส้นโค้ง $y = x^2 - 4x + 4$ ที่มีระยะห่างจากจุด origin น้อยที่สุด

$y = (x-2)^2$



$2x^3 - 12x^2 + 25x - 16 = 0$