# On two families of generalizations of Pascal's triangle

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## Abstract

We consider two families of Pascal-like triangles that have all ones on the left side and ones separated by $m-1$ zeros on the right side. The $m=1$ cases are Pascal's triangle and the two families also coincide when $m=2$. Members of the first family obey Pascal's recurrence everywhere inside the triangle. We show that the $m$-th triangle can also be obtained by reversing the elements up to and including the main diagonal in each row of the $(1/(1-x^m), x/(1-x))$ Riordan array. Properties of this family of triangles can be obtained quickly as a result. The $(n,k)$-th entry in the $m$-th member of the second family of triangles is the number of tilings of an $(n+k)\times 1$ board that use $k$ $(1,m-1)$-fences and $n-k$ unit squares. A $(1,g)$-fence is composed of two unit square sub-tiles separated by a gap of width $g$. We show that the entries in the antidiagonals of these triangles are coefficients of products of powers of two consecutive Fibonacci polynomials and give a bijective proof that these coefficients give the number of $k$-subsets of $\{1,2,\ldots,n-m\}$ such that no two elements of a subset differ by $m$. Other properties of the second family of triangles are also obtained via a combinatorial approach. Finally, we give necessary and sufficient conditions for any Pascal-like triangle (or its row-reversed version) derived from tiling $(n\times 1)$-boards to be a Riordan array.


| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | **1** | | | | | | | | | | | | | |
| 1 | **1** | **0** | | | | | | | | | | | | |
| 2 | **1** | **0** | **0** | | | | | | | | | | | |
| 3 | **1** | **1** | **1** | **1** | | | | | | | | | | |
| 4 | **1** | **2** | **3** | **2** | **0** | | | | | | | | | |
| 5 | **1** | **3** | 5 | **4** | **0** | **0** | | | | | | | | |
| 6 | **1** | **4** | 8 | **8** | **4** | **2** | **1** | | | | | | | |
| 7 | **1** | **5** | 12 | 16 | 13 | **9** | **3** | **0** | | | | | | |
| 8 | **1** | **6** | 17 | 28 | **30** | 22 | **9** | **0** | **0** | | | | | |
| 9 | **1** | **7** | 23 | 45 | 58 | 51 | **27** | **9** | **3** | **1** | | | | |
| 10 | **1** | **8** | 30 | 68 | 103 | 108 | 78 | 40 | **18** | **4** | **0** | | | |
| 11 | **1** | **9** | 38 | 98 | 171 | 211 | 187 | **123** | 58 | **16** | **0** | **0** | | |
| 12 | **1** | **10** | 47 | 136 | 269 | 382 | 399 | 310 | 176 | **64** | **16** | **4** | **1** | |
| 13 | **1** | **11** | 57 | 183 | 405 | 651 | 781 | 708 | 480 | 240 | 90 | **30** | **5** | **0** |

The start of the $m=3$ member of the $n$-tile tiling-derived Pascal-like triangle family (A350110 in the OEIS). Entries in bold font are covered by identities derived in the paper.

## Background

Pascal's triangle is familiar to most people; written as an infinite lower triangular matrix, we fill the leftmost vertical column and the leading diagonal with ones and then use 'Pascal's recurrence' (an entry equals the sum of the entry above and the entry above and to the left) to fill in the rest. Pascal's triangle has many interesting properties and a myriad of applications in combinatorics.

What happens if we replace the leading diagonal by the periodic pattern of 1 followed by $m-1$ zeros for $m=1,2,\ldots$ while still using Pascal's recurrence to fill in the other entries? The paper shows that the family of triangles obtained are also a family of row-reversed Riordan arrays. A $(p(x),q(x))$ Riordan array is an infinite lower triangular matrix whose $(n,k)$-th entry (with row number $n$ and column number $k$ both starting at zero) is the coefficient of $x^n$ in the series expansion of $p(x)(q(x))^k$. For example, Pascal's triangle is the $(1/(1-x), x/(1-x))$ Riordan array. Riordan arrays have numerous applications in combinatorics such as counting walks on lattices. Generating functions for various sums of entries in a Riordan array can be obtained quickly in terms of $p$ and $q$.

The $(n,k)$-th entry of Pascal's triangle can be obtained in other ways: it is ${}^nC_k$, the number of ways of choosing $k$ objects from $n$. It is also is the number of square-and-domino tilings of $N$-boards (which are linear arrays $N$ unit square cells) that use $n$ tiles in total of which $k$ are $(2 \times 1)$ dominoes (and therefore $n-k$ are unit squares). This is easily seen since there are ${}^nC_k$ ways to choose which $k$ of the $n$ tiles are dominoes.

What triangles do we get if we instead tile using squares and 'split dominoes'? A split domino has its two halves separated by a gap of width $m-1$ and is known as a $(1, m-1)$-fence. It turns out that the family of triangles so generated have the same boundaries as the other family of triangles we consider and the other entries are identical in the $m=1$ and $m=2$ cases.

A question which we show can be answered in terms of tilings with squares and $k$ $(1, m-1)$-fences is how many subsets with $k$ elements can be chosen from the numbers $1, \ldots, n$ such that no two elements in the subset differ by $m$. E.g., if $m=3$, the possible subsets of $\{1,2,3,4,5\}$ are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1,2\}$, $\{1,3\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, $\{3,5\}$, $\{4,5\}$, $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$, and $\{1,3,5\}$. The numbers of subsets are entries in the tiling-derived family of triangles we consider. E.g., the 8th antidiagonal of the $m=3$ triangle is 1,5,8,4 which are the numbers of subsets of $\{1,2,3,4,5\}$ of sizes 0, 1, 2, and 3, respectively.

## Key results

- Expression for $(n,k)$-th entry in the Riordan array triangles as a sum of binomial coefficients.
- Generating functions for subdiagonals, row sums, antidiagonal sums, and for the whole triangle in the case of the Riordan array triangles.
- Relation between tiling triangles and certain classes of polynomials.
- Expression for entry in the tiling triangles in terms of the coefficient of the products of powers of two consecutive Fibonacci polynomials.
- Expression for antidiagonal sums of the tiling triangles as products of powers of two consecutive Fibonacci numbers.
- Bijection between the $k$-subsets of $\{1, \ldots, n\}$ such that all pairs of elements taken from a subset do not differ by $m$, and the tilings of an $(n+m)$-board with $k$ $(1, m-1)$-fences and $n+m-2k$ squares. The number of such subsets is the $(n+m-k, k)$-th entry of the $m$-th tiling triangle.
- 7 identities and 1 conjecture concerning the tiling triangles in general.
- 3 additional identities concerning the $m=3$ tiling triangle.
- Conditions for a tiling-derived Pascal-like triangle or its row-reversed version to be a Riordan array.
- Generating functions for row-reversed Riordan arrays and the sums of their antidiagonals.

## Related resources


[1] Sloane NJA (2022) *The On-Line Encyclopedia of Integer Sequences*, `oeis.org`.
[2] Sprugnoli R (1994) Riordan arrays and combinatorial sums. *Discrete Math* **132**, 267–290.
[3] Allen MA (2019) *Riordan Arrays* seminar `www.youtube.com/watch?v=qMhSxcwlHvM`.
[4] Edwards K, Allen MA (2021) New combinatorial interpretations of the Fibonacci numbers squared, golden rectangle numbers, and Jacobsthal numbers using two types of tile. *J Integer Sequences* **24**, 21.3.8.
[5] Allen MA (2022) On a two-parameter family of generalizations of Pascal's triangle. *J Integer Sequences* **25**, 22.9.8.