Fence tiling derived identities involving the metallonacci numbers squared or cubed

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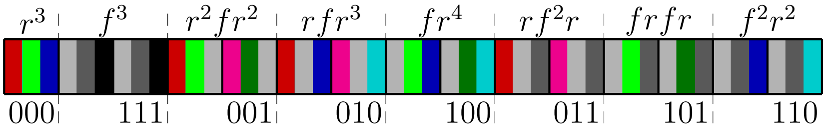
Abstract

We refer to the generalized Fibonacci sequence $(M_n^{(c)})_{n\geq 0}$, where $M_{n+1}^{(c)} = cM_n^{(c)} + M_{n-1}^{(c)}$ for $n > 0$ with $M_0^{(c)} = 0$, $M_1^{(c)} = 1$, for $c = 1, 2, \ldots$ as the c-metallonacci numbers. We consider the tiling of an n-board (an $n \times 1$ rectangular board) with c colours of $1/p \times 1$ tiles (with the shorter sides always aligned horizontally) and $(1/p, 1-1/p)$-fence tiles for $p \in \mathbb{Z}^+$. A (w, g)-fence tile is composed of two $w \times 1$ sub-tiles separated by a $g \times 1$ gap. The number of such tilings equals $(M_{n+1}^{(c)})^p$ and we use this result for the cases $p = 2, 3$ to devise straightforward combinatorial proofs of identities relating the metallonacci numbers squared or cubed to other combinations of metallonacci numbers. Special cases include relations between the Pell numbers cubed and the even Fibonacci numbers. Most of the identities derived here appear to be new.



A 15-board tiled with metatiles of length less than 3 when $p = 3$. Symbolic representations are shown above the metatiles and final cell slot contents σ (with a 0 representing a rectangular tile r and 1 representing the right sub-tile of a fence f) are given below. Dashed lines show boundaries between metatiles.

Background

It is well known that the ratio F_{n+1}/F_n of two successive Fibonacci numbers (where $F_n = F_{n-1}+F_{n-2}+\delta_{n,1}$, $F_{n<1} = 0$, and $\delta_{i,j}$ is 1 if $i = j$ and zero otherwise) tends to the golden ratio, $(1+\sqrt{5})/2$, as $n \to \infty$. The golden ratio is the $c = 1$ case of the continued fraction

$$\phi^{(c)} = c + \cfrac{1}{c + \cfrac{1}{c + \cfrac{1}{c + \cfrac{1}{c + \ddots}}}} = \frac{c + \sqrt{c^2+4}}{2}.$$

$\phi^{(2)}$ is called the silver ratio and is the limit as $n \to \infty$ of P_{n+1}/P_n, where the Pell numbers $P_n = 2P_{n-1} + P_{n-2} + \delta_{n,1}$, $P_{n<1} = 0$. In general, the continued fractions $\phi^{(c)}$ for positive integer c are called the metallic ratios and $\phi^{(c)} = M_{n+1}^{(c)}/M_n^{(c)}$ as $n \to \infty$ where $M_n^{(c)} = cM_{n-1}^{(c)} + M_{n-2}^{(c)} + \delta_{n,1}$, $M_{n<1}^{(c)} = 0$. It is for this reason that we refer to the generalized Fibonacci sequence $(M_n^{(c)})_{n\geq 0} = 0, 1, c, c^2+1, \ldots$ as the c-metallonacci numbers.

Enumerating tilings of finite boards can give a physical picture of various integer sequences. The number of ways to tile an n-board (which is a $1 \times n$ array of unit square cells) using unit squares and dominoes (two unit squares joined at one edge) is F_{n+1}. If there are c possible colours of square then the number of ways to tile an n-board is $M_{n+1}^{(c)}$. The number of ways to tile p n-boards using such tiles is therefore $(M_{n+1}^{(c)})^p$ and this is the simplest combinatorial interpretation of the metallonacci numbers raised to a positive integer power. Combinatorial interpretations such as these can be the basis for quick intuitive proofs of identities instead of using algebra.

Is there a combinatorial interpretation of $(M_{n+1}^{(c)})^p$ by considering the tiling of a single n-board when $p = 2, 3, \ldots$? The answer is yes if we tile with $1/p \times 1$ tiles (denoted by r) with the shorter side always aligned horizontally and so-called $(1/p, 1-1/p)$-fence tiles (denoted by f). The latter are tiles composed of two $1/p \times 1$ sub-tiles separated by a gap of $1 - 1/p$.

A tiling which includes fractional length tiles such as rectangular or fence tiles can be reduced to a tiling using metatiles. A metatile is a grouping of tiles that exactly covers an integer number of cells and cannot be split into smaller metatiles. Evaluating the number of metatiles of a given length is the key to obtaining convolution-type identities via this class of combinatorial interpretation. Although there are infinite number of possible metatiles when $p > 1$, the evaluation is straightforward in the $p = 2$ case. However, in the $p = 3$ case we have only been able to obtain an expression for the number of metatiles of a given length by first obtaining recursion relations for the number of metatiles of a given length with a particular ending σ and then combining the results.

Key results

- A name for a particular class of generalized Fibonacci numbers.
- Two new combinatorial interpretations of $(M_n^{(c)})^p$ for $p = 2, 3, \ldots$.
- 6 new identities involving $(M_n^{(c)})^2$.
- 7 new identities involving $(M_n^{(c)})^3$ including one which in the $c = 2$ case is

$$P_{n+1}P_nP_{n-1} = 5\sum_{k=1}^{n-1} F_{3k}P_{n-k}^3.$$

- Pascal-like triangle (which is also a row-reversed Riordan array) whose n-th row sums to give $(M_{n+1}^{(c)})^2$.

Related resources

[1] Allen MA (2022) *Fence tiling derived identities involving the metallonacci numbers squared or cubed* talk www.youtube.com/watch?v=IYS7IdTN1VY.
[2] en.wikipedia.org/wiki/Metallic_mean.
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[6] Edwards K, Allen MA (2019) A new combinatorial interpretation of the Fibonacci numbers squared. *Fibonacci Quart* **57(5)**, 48–53.
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[10] Sprugnoli R (1994) Riordan arrays and combinatorial sums. *Discrete Math* **132**, 267–290.
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[12] Allen MA, Edwards K (2022) On a two-parameter family of generalizations of Pascal's triangle. *J Integer Sequences* **25**, 22.7.1.