Identities relating permanents of some classes of (0,1) Toeplitz matrices to generalized Fibonacci numbers

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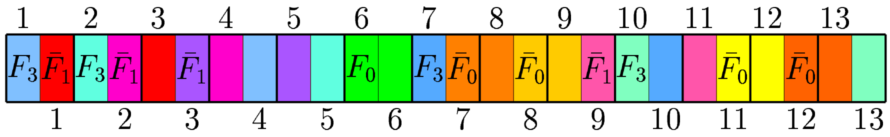
Abstract

We combinatorially prove identities relating the permanents of various classes of (0,1) Toeplitz matrices to some sequences generated from linear homogeneous finite-order recursion relations with positive integer coefficients and integer-valued initial conditions. This is done using a previously obtained bijection between permanents of (0,1) Toeplitz matrices and the tilings of an $n \times 1$ board with $(\frac{1}{2}, w)$-fences, where w is a nonnegative integer. A $(\frac{1}{2}, w)$-fence is a tile composed of two $\frac{1}{2} \times 1$ rectangular sub-tiles aligned horizontally and separated by a gap of width w.



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 4 & 5 & 1 & 2 & 3 & 6 & 10 & 7 & 8 & 13 & 9 & 11 & 12 \end{pmatrix}$$

A tiling of a 13-board with down F_1, down F_0, up F_0, and up F_3 (and so $\{-2, -1, 0, 3\} \subseteq D$) and the corresponding permutation. The left post of each fence is labelled. A bar over the F denotes a down fence. No bar indicates an up fence.

Background

The *permanent* of an $n \times n$ matrix M, whose (i, j)-th entry is denoted by $M_{i,j}$, is the same as the determinant but with all plus signs. Permanents of (0,1) $n \times n$ matrices (i.e., $n \times n$ matrices whose elements are 0 or 1) give the number of permutations $\pi(i)$ of $i \in \mathbb{N}_n = \{1, 2, \ldots, n\}$. We can view the entries in M as detailing which permutations are allowed: if $M_{i,j} = 1$, then i can get mapped to j (i.e., $\pi(i) = j$ is allowed); otherwise, if $M_{i,j} = 0$, then $\pi(i)$ cannot equal j. Here we consider restricted permutations where the only allowed permutations are such that $\pi(i) - i \in D$, where D is a finite set. M is then a *Toeplitz matrix*. This is a matrix with constant diagonals in the sense that $M_{i+1,j+1} = M_{i,j}$ for all allowed i and j. Permanents of such matrices also have a combinatorial interpretation in terms of a class of tilings of an n-board with $(1/2, w)$-fences (denoted by F_w) for selected nonnegative integers w. The two sub-tiles of a fence are referred to as *posts*. As illustrated in the figure, an up fence has its left post in the left side of a cell on the board and corresponds to a positive $\pi(i) - i$; a down fence has its right post in the left side of a cell and corresponds to a negative $\pi(i) - i$. The $\pi(i) = i$ case correponds to a fence with no gap (an F_0) aligned with a cell on the board. The combinatorial interpretation allows one to prove identities involving permanents using a combinatorial rather than algebraic approach by counting the number of possible tilings using the fences corresponding to the elements in D.

The Fibonacci numbers $\{f_n\}_{n \geq 0}$ are defined by $f_n = f_{n-1} + f_{n-2} + \delta_{n,0}$, $f_{n<0} = 0$, where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise. We refer to a sequence of numbers defined by an analogous recursion relation with arbitrarily many terms on the right-hand side but all with positive integer coefficients (the coefficients of the δ's can be integers of either sign; these define the boundary conditions) as generalized Fibonacci numbers.

Key results

- An expression relating permanents of odd dimension (0,1) Toeplitz matrices with $D = \{-1, 0, d_1, ..., d_k\}$ where the d_i are odd, positive, and distinct to permanents of even dimension Toeplitz matrices of the same type and generalized Fibonacci numbers.
- A theorem that can be used to generate identities relating permanents of (0,1) Toeplitz matrices to generalized Fibonacci numbers for 24 instances of D.

- 7 new identities relating permanents of $n \times n$ (0,1) Toeplitz matrices with a particular D to various types of generalized Fibonacci numbers.

Related resources

[1] Allen MA (2024) *Identities relating permanents of some classes of (0,1) Toeplitz matrices to generalized Fibonacci numbers* talk `www.youtube.com/watch?v=EAz28aafQJs`.

[2] Edwards K (2008/2009) A Pascal-like triangle related to the tribonacci numbers. *Fibonacci Quart* **46/47(1)**, 18–25.

[3] Edwards K, Allen MA (2015) Strongly restricted permutations and tiling with fences. *Discrete Appl Math* **187**, 82–90.

[4] Benjamin AT, Quinn JJ (2003) *Proofs That Really Count: The Art of Combinatorial Proof*, Mathematical Association of America.