On a two-parameter family of generalizations of Pascal's triangle

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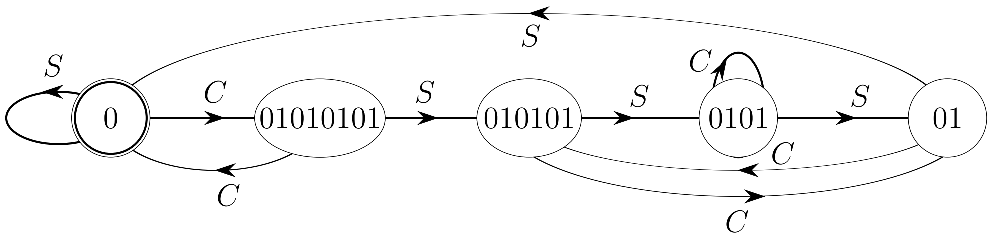
Abstract

We consider a two-parameter family of triangles whose (n,k)-th entry (counting the initial entry as the $(0,0)$-th entry) is the number of tilings of N-boards (which are linear arrays of N unit square cells for any nonnegative integer N) with unit squares and $(1,m-1;t)$-combs for some fixed $m=1,2,\dots$ and $t=2,3,\dots$ that use n tiles in total of which k are combs. A $(1,m-1;t)$-comb is a tile composed of t unit square sub-tiles (referred to as teeth) placed so that each tooth is separated from the next by a gap of width $m-1$. We show that the entries in the triangle are coefficients of the product of two consecutive generalized Fibonacci polynomials each raised to some nonnegative integer power. We also present a bijection between the tiling of an $(n+(t-1)m)$-board with k $(1,m-1;t)$-combs with the remaining cells filled with squares and the k-subsets of $\{1,\dots,n\}$ such that no two elements of the subset differ by a multiple of m up to $(t-1)m$. We can therefore give a combinatorial proof of how the number of such k-subsets is related to the coefficient of a polynomial. We also derive a recursion relation for the number of closed walks from a particular node on a class of directed pseudographs and apply it obtain an identity concerning the $m=2$, $t=5$ instance of the family of triangles. Further identities of the triangles are also established mostly via combinatorial proof.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	**1**													
1	**1**	**0**												
2	**1**	**0**	**1**											
3	**1**	**0**	**2**	**0**										
4	**1**	**0**	**4**	**0**	**1**									
5	**1**	**1**	**6**	**0**	**3**	**0**								
6	**1**	**2**	**9**	**0**	**9**	**0**	**1**							
7	**1**	**3**	**12**	**5**	**18**	**0**	**4**	**0**						
8	**1**	**4**	**16**	12	**36**	**0**	**16**	**0**	**1**					
9	**1**	**5**	**20**	25	**60**	**15**	**40**	**0**	**5**	**0**				
10	**1**	**6**	**25**	42	**100**	42	**100**	**0**	**25**	**0**	**1**			
11	**1**	**7**	31	66	**150**	112	**200**	**35**	**75**	**0**	**6**	**0**		
12	**1**	**8**	38	96	**225**	224	**400**	112	**225**	**0**	**36**	**0**	**1**	
13	**1**	**9**	46	134	325	424	**700**	364	**525**	**70**	**126**	**0**	**7**	**0**



The table shows the start of the $m=2$, $t=5$ member of the Pascal-like triangle family (A354667 in the OEIS). The (n,k)-th entry is the number of tilings of N-boards that use k $(1,1;5)$-combs and $n-k$ squares. Entries in bold font are covered by identities derived in the paper. Also shown is the digraph for generating metatiles when tiling with squares and $(1,1;5)$-combs.

Background

The (n,k)-th entry of Pascal's triangle is ${}^nC_k$, the number of ways of choosing k objects from n. It is also is the number of square-and-t-omino tilings of N-boards (which are linear arrays N unit square cells) that use n tiles in total of which k are t-ominoes (tiles with dimensions $1\times t$) and therefore $n-k$ are unit squares. This is easily seen since there are ${}^nC_k$ ways to choose which k of the n tiles are t-ominoes.

What triangles do we get if we instead tile using squares and 'split t-ominoes'? A split t-omino is separated into t squares each separated from the next by a gap of width $m-1$ and is known as a $(1, m-1; t)$-comb. It is these triangles that we investigate in this work which follows on from [1] in which the second family of one-parameter generalizations of Pascal's triangle considered corresponds to tiling with squares and $(1, m-1; 2)$-combs (also known as $(1, m-1)$-fences).

A question which we show can be answered in terms of tilings with squares and k $(1, m-1; t)$-combs is how many subsets with k elements can be chosen from the numbers $1, \dots, n$ such that no two elements in the subset differ by a multiple of m up to $(t-1)m$. E.g., if $m = 2$ and $t = 5$, so that no two elements in the subset differ by 2, 4, 6, or 8, then the possible subsets of $\{1, 2, 3, 4, 5\}$ are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1, 2\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 5\}$, $\{3, 4\}$, and $\{4, 5\}$. The numbers of subsets are entries in the tiling-derived family of triangles we consider. E.g., the 13th (1,4)-antidiagonal of the $m = 2$, $t = 5$ triangle is 1,5,6 which are the numbers of subsets of $\{1, 2, 3, 4, 5\}$ obeying the given restrictions of sizes 0, 1, and 2, respectively.

To obtain results concerning the triangles we need to consider the metatiles involved in the tiling. A metatile is a grouping of tiles that exactly covers an integer number of cells with no gaps and cannot be split into smaller metatiles. Simple examples of metatiles are a single square (S), a comb (C) with all the gaps filled with squares (which we denote by $CS^{(m-1)(t-1)}$ since $m-1$ squares fit in each of the $t-1$ gaps of the comb), and m interlocking combs (C^m). All tilings of N-boards are concatenations of one or more metatiles.

We need to find a recursion relation for the number of tilings of N-boards that contain n tiles of which k are combs. This is done with the help of a metatile-generating digraph. Each path beginning and ending on the 0 node without visiting it in between corresponds to a metatile. The 0 node represents the empty board or the completed metatile. Each of the other nodes gives the state of the yet-to-be-completed metatile with 0 (1) representing an empty (filled) cell. The arcs in the digraph correspond to the addition of a tile. The first step in deriving the recursion relation is to count the number of paths of length n that start and finish at the 0 node without visiting it in between; this is the number of metatiles containing n tiles.

Key results

- Expression for each entry in the triangles in terms of the coefficient of the products of powers of two consecutive generalized Fibonacci polynomials.

- Expression for sums of elements of the triangles along particular rays as products of powers of two consecutive generalized Fibonacci numbers.

- Bijection between the k-subsets of $\{1, \dots, n\}$ such that all pairs of elements taken from a subset do not differ by a multiple of m up to $(t-1)m$, and the tilings of an $(n+(t-1)m)$-board with k $(1, m-1; t)$-combs and $n + (t-1)m - kt$ squares. The number of such subsets is the $(n + (t-1)(m-k), k)$-th entry of the triangle.

- 9 identities and 1 theorem concerning the tiling triangles in general.

- 12 additional identities concerning particular instances of the triangles.

- Recursion relation for 3-inner cycle digraphs with a pseudo-common node (an example of which is shown in the figure).

Related resources

[1] Allen MA, Edwards K (2022) On a two-parameter family of generalizations of Pascal's triangle. *J Integer Sequences* **25**, 22.7.1.

[2] Allen MA, Edwards K (2022) Connections between two classes of generalized Fibonacci numbers squared and permanents of (0,1) Toeplitz matrices. *Linear Multilinear Algebra*, `https://doi.org/10.1080/03081087.2022.2107979`, `https://arxiv.org/abs/2107.02589`.

[3] Edwards K, Allen MA (2015) Strongly restricted permutations and tiling with fences. *Discrete Appl. Math.* **187**, 82–90.

[4] Sloane NJA (2022) *The On-Line Encyclopedia of Integer Sequences*, `oeis.org`.