Connections between combinations without specified separations and strongly restricted permutations, compositions, and bit strings

Michael A. Allen*

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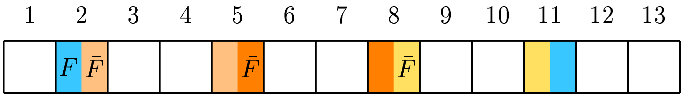
Abstract

Let S_n and $S_{n,k}$ be, respectively, the number of subsets and k-subsets of $\mathbb{N}_n = \{1, \ldots, n\}$ such that no two subset elements differ by an element of the set Q, the largest element of which is q. We prove a bijection between such k-subsets when $Q = \{m, 2m, \ldots, jm\}$ with $j, m > 0$ and permutations π of $\mathbb{N}_{n+jm}$ with k excedances satisfying $\pi(i) - i \in \{-m, 0, jm\}$ for all $i \in \mathbb{N}_{n+jm}$. We also identify a bijection between another class of restricted permutation and the cases $Q = \{1, q\}$ and derive the generating functions for S_n when $q = 4, 5, 6$. We give some classes of Q for which S_n is also the number of compositions of $n + q$ into a given set of allowed parts. We also prove a bijection between k-subsets for a class of Q and the set representations of size k of equivalence classes for the occurrence of a given length-$(q + 1)$ subword within bit strings. We then formulate a straightforward procedure for obtaining the generating function for the number of such equivalence classes.



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 1 & 11 & 3 & 4 & 2 & 6 & 7 & 5 & 9 & 10 & 8 & 12 & 13 \end{pmatrix}$$

A 13-board tiled with an up $(\frac{1}{2}, 9)$-fence (F; blue), three down $(\frac{1}{2}, 2)$-fences (F-bar; yellow, orange, and ochre), and 9 squares, and the corresponding example of a $D = \{-3, 0, 9\}$ restricted permutation.

Background

An ordinary combination refers to choosing any k objects from n objects, which we can take as the numbers $\mathbb{N}_n = \{1, \ldots, n\}$. A subset containing k objects is called a k-subset. The number of k-subsets of $\mathbb{N}_n$ is well known to be ${}^nC_k$. Combinations without specified separations (also called restricted combinations) refers to choosing k-subsets such that no two elements of the subset differ by an element of the set Q, the largest member of which is q. We let $S_{n,k}$ denote the number of such subsets and S_n the number of all subsets of $\mathbb{N}_n$ that satisfy the disallowed separations requirement. For example, if $Q = \{1\}$ then the allowed subsets of $\mathbb{N}_4 = \{1, 2, 3, 4\}$ are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 3\}$, $\{1, 4\}$, and $\{2, 4\}$, and so $S_{4,0} = 1$, $S_{4,1} = 4$, $S_{4,2} = 3$, and $S_4 = 8$ in this case. In fact, when $Q = \{1\}$, it is a classic result that S_n is the Fibonacci number f_{n+1} (where $f_n = f_{n-1} + f_{n-2} + \delta_{0,n}$, $f_{n<0} = 0$), and $S_{n,k} = {}^{n+1-k}C_k$. Up until recently, the only results for S_n and $S_{n,k}$ known were the general case $Q = \{m, 2m, \ldots, jm\}$ with $j, m > 0$ along with another class of Q (which, among other properties, has $1 \in Q$). Then the author came up with a 1-1 correspondence between restricted combinations of $\mathbb{N}_n$ and a form of tiling of an $(n + q)$-board (an $(n + q) \times 1$ board consisting of 1×1 cells). This bijection allows other results for various other classes of Q to be obtained easily, and in the present paper the author explains how this can be used to obtain the generating functions for S_n and $S_{n,k}$ for any Q. The coefficients of x^n and $x^n y^k$ in the expansions of generating functions $g(x)$ and $g(x, y)$ equal S_n and $S_{n,k}$, respectively. For example, the generating functions for the $Q = \{1\}$ case are $g(x) = 1/(1 - x - x^2) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots$ and $g(x, y) = 1/(1 - x - x^2 y) = 1 + x + x^2(1 + y) + x^3(1 + 2y) + x^4(1 + 3y + y^2) + x^5(1 + 4y + 3y^2) + \cdots$. The n-board is restricted-overlap tiled with 1×1 squares and Q-combs. In general a Q-comb is composed of a number of sub-tiles (called teeth) which, if there is more than one tooth, are separated by gaps. The cells of the Q-comb (whether occupied by a tooth or not) are numbered from 0 to q. Cell 0 of a Q-comb is always occupied by the (start of) the leftmost tooth of the comb. Cell i, where $i = 1, \ldots, q$, is occupied by a tooth if and only if i is in Q. Restricted-overlap tiling means that all but cell 0 of a Q-comb can overlap all but cell 0 of one or more other Q-combs already placed on the board. After all the combs have been placed on the board, any remaining gaps are filled with squares.

The author noticed that S_n for various Q matched existing sequences in the On-Line Encyclopedia of Integer Sequences. One of these connections is illustrated in the figure. A $(\frac{1}{2}, w)$-fence is a tile composed of two sub-tiles each of width $\frac{1}{2}$ and separated by a gap of w. They, along with squares, can be used to represent a particular type of restricted permutation characterized by a set D of allowed displacements of the items. A fence representing an upward displacement (known as an excedance) is called an up fence and must be placed so that the left sub-tile is in the left cell side of a cell. A fence representing a downward displacement must be placed so that its right sub-tile is in the left side of a cell on the board. When placed as illustrated in the figure, the fences form a Q-comb (in the figure, the Q-comb correponding to $Q = \{3, 6, 9\}$). This forms the basis for the connection between a class of restricted permutation and a class of restricted combination.

Key results

- Bivariate generating functions for the number of walks of a certain length and containing a certain number of objects on two general classes of digraph.
- General procedures for obtaining bivariate generating functions from digraphs with and without the use of transfer matrices.
- A bijection between the cases $Q = \{1, q\}$ and a type of restricted permutation.
- Generating functions for S_n when $Q = \{1, q\}$ for $q = 4, 5, 6$.
- A bijection between the cases $Q = \{m, 2m, \ldots, jm\}$ and restricted permutations with $D = \{-m, 0, jm\}$.
- Expressions for numbers of restricted permutations with $D = \{-m, 0, jm\}$ and numbers of such permutations with k excedances in terms of generalized Fibonacci numbers and coefficients of their associated polynomials.
- Condition on Q for S_n to equal the number of compositions of $n + q$ into parts drawn from a finite set and two classes of Q for which this condition holds.
- Necessary conditions on digraph corresponding to Q for S_n to equal the number of compositions of $n + q$ into parts drawn from an infinite set and two classes of Q where S_n does equal the number of such compositions.
- Bijection between subsets of N_{n-q} for certain Q and the equivalence classes of the appearance of length-$(q + 1)$ subword ω in length-n binary words and the associated generating functions.
- Condition on Q for there to exist a corresponding ω.
- An algorithm for efficiently calculating S_n and $S_{n,k}$ (from first principles) for n up to 32 or 64 for any given Q.

Related resources

[1] Allen MA (2024) Combinations without specified separations. *Comm Combinator Optim* (in press) `doi.org/10.22049/CCO.2024.29370.1959`.
[2] Edwards K (2008/2009) A Pascal-like triangle related to the tribonacci numbers. *Fibonacci Quart* **46/47(1)**, 18–25.
[3] Edwards K, Allen MA (2015) Strongly restricted permutations and tiling with fences. *Discrete Appl. Math.* **187**, 82–90.
[4] Sloane NJA (2025) *The On-Line Encyclopedia of Integer Sequences*, `oeis.org`.