New combinatorial interpretations of the Fibonacci numbers squared, golden rectangle numbers, and Jacobsthal numbers using two types of tile

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Abstract

We consider the tiling of an n-board (a board of size $n \times 1$) with squares of unit width and $(1, 1)$-fence tiles. A $(1, 1)$-fence tile is composed of two unit-width square sub-tiles separated by a gap of unit width. We show that the number of ways to tile an n-board using unit-width squares and $(1, 1)$-fence tiles is equal to a Fibonacci number squared when n is even and a golden rectangle number (the product of two consecutive Fibonacci numbers) when n is odd. We also show that the number of tilings of boards using n such square and fence tiles is a Jacobsthal number. Using combinatorial techniques we prove new identities involving golden rectangle and Jacobsthal numbers. Two of the identities involve entries in two Pascal-like triangles. One is a known triangle (with alternating ones and zeros along one side) whose (n, k)th entry is the number of tilings using n tiles of which k are fence tiles. There is a simple relation between this triangle and the other which is the analogous triangle for tilings of an n-board. These triangles are related to Riordan arrays and we give a general procedure for finding which Riordan array(s) a triangle is related to. The resulting combinatorial interpretation of the Riordan arrays allows one to derive properties of them via combinatorial proof.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	0								
2	1	1	1							
3	1	2	2	0						
4	1	3	4	2	1					
5	1	4	7	6	3	0				
6	1	5	11	13	9	3	1			
7	1	6	16	24	22	12	4	0		
8	1	7	22	40	46	34	16	4	1	
9	1	8	29	62	86	80	50	20	5	0

The start of a Pascal-like triangle (A059259 in the OEIS) whose (n, k)th entry is the number of tilings of a board using k $(1, 1)$-fences and $n - k$ squares. Reversing the rows gives the $(1/(1 - x^2), x/(1 - x))$ Riordan array.

Background

Enumerating tilings of finite boards can give a physical picture of various integer sequences. For example, the number of ways to tile an n-board (a linear array of n square cells) with squares and dominoes is the Fibonacci number F_{n+1}, where $F_n = F_{n-1} + F_{n-2} + \delta_{n,1}$, $F_{n<1} = 0$; the number of ways to tile an n-board with squares of one colour and two different colours of domino is the Jacobsthal number J_{n+1}, where $J_n = J_{n-1} + 2J_{n-2} + \delta_{n,1}$, $J_{n<1} = 0$. Combinatorial interpretations such as these can be used to give quick intuitive proofs of identities instead of using algebra.

A $(p(x), q(x))$ Riordan array, where $p(x) = p_0 + p_1x + p_2x^2 + \cdots$ and $q(x) = q_1x + q_2x^2 + \cdots$, is an infinite lower triangular matrix whose (n, k)th entry is the coefficient of x^n in the series for $p(x)\{q(x)\}^k$. E.g., Pascal's triangle is the $(1/(1 - x), x/(1 - x))$ Riordan array. Such arrays have many applications in combinatorics.

Key results

- A new combinatorial interpretation of $F_n^{m-r}F_{n+1}^r$ for $r = 0, \ldots, m - 1$ and $m = 2, 3, \ldots$.
- A new combinatorial interpretation of the Jacobsthal numbers, J_n.
- 2 new identities concerning the golden rectangle numbers, F_nF_{n+1}.

- 4 new identities concerning J_n. E.g., $J_{n+1} = \lceil (n+1)/2 \rceil + \sum_{j=1}^{n-1} j J_{n-j}$.
- A procedure for determining what type of Riordan array(s) a tiling-derived Pascal-like triangle corresponds to.
- Combinatorial proofs of properties of a known Pascal-like triangle which is also a row-reversed Riordan array. These include a proof of a conjecture about the array.
- Description and properties of a new Pascal-like triangle (A335964 in the OEIS).

Related resources

[1] Sloane NJA (2010) *The Online Encyclopedia of Integer Sequences*, `oeis.org`.
[2] Benjamin AT, Quinn JJ (2003) *Proofs That Really Count: The Art of Combinatorial Proof*, Mathematical Association of America.
[3] Sprugnoli R (1994) Riordan arrays and combinatorial sums. *Discrete Math* **132**, 267–290.
[4] Allen MA (2019) *Riordan Arrays* seminar `www.youtube.com/watch?v=qMhSxcwlHvM`.
[5] Edwards K (2008/2009) A Pascal-like triangle related to the tribonacci numbers. *Fibonacci Quart* **46/47(1)**, 18–25.
[6] Edwards K, Allen MA (2020) A new combinatorial interpretation of the Fibonacci numbers squared. Part II. *Fibonacci Quart* **58(2)**, 169–177.