Lecture 4: Simple Harmonic Oscillator and Coherent States

Kit Matan

Mahidol University

4.1 Operator method

For the simple harmonic oscillator in one dimension, the Hamiltonian is given by

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2,$$

where P and X are the momentum and position operators, respectively. We will define two new operators called a and $a^\dagger$ in terms of X and P:

$$\begin{aligned} a &= \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{iP}{m\omega}\right) \\ a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right) \end{aligned}$$

Note that a is not Hermitian ($a \neq a^\dagger$) and hence not an observable. Next we will consider

$$\begin{aligned} a^\dagger a &= \frac{m\omega}{2\hbar}X^2 + \frac{P^2}{2\hbar m\omega} + \frac{i}{2\hbar}[X,P] \\ &= \frac{1}{\hbar\omega}\left(\frac{P^2}{2m} + \frac{m\omega^2}{2}X^2\right) + \frac{i}{2\hbar}(i\hbar), \quad \text{where } [X,P] = i\hbar \\ &= \frac{H}{\hbar\omega} - \frac{1}{2} \\ \Rightarrow H &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \end{aligned} \tag{4.1}$$

The commutator relation between a and $a^\dagger$ can be calculated using $[X,P] = i\hbar$:

$$[a,a^\dagger] = aa^\dagger - a^\dagger a = \frac{m\omega}{2\hbar}X^2 + \frac{P^2}{2\hbar m\omega} + \frac{i}{2\hbar}[P,X] - \frac{m\omega}{2\hbar}X^2 - \frac{P^2}{2\hbar m\omega} - \frac{i}{2\hbar}[X,P] = 1$$

We will define $N \equiv a^\dagger a$ and call it a "number operator".

$$\Rightarrow H = \hbar\omega\left(N + \frac{1}{2}\right)$$

In this sense, N is only different from H by a constant, in particular $[H,N] = 0$, and hence we can label eigenstates of H using eigenvalues of N. Suppose that n and $|n\rangle$ are eigenvalues and eigenstate of N, respectively, such that

$$N|n\rangle = n|n\rangle.$$

Now consider

$$\begin{aligned} Na\left|n\right\rangle &= a^\dagger aa\left|n\right\rangle \\ &= \left(aa^\dagger-1\right)a\left|n\right\rangle, \quad \text{since } a^\dagger a = aa^\dagger - 1 \\ &= a\left(a^\dagger a-1\right)\left|n\right\rangle \\ &= (n-1)\left[a\left|n\right\rangle\right] \end{aligned}$$

Therefore, the sate $a\left|n\right\rangle$ is the same state as $\left|n-1\right\rangle$ since they both have the same eigenvalues for N, that is,

$$\left|n-1\right\rangle = C_{n-1}a\left|n\right\rangle.$$

In other words, the operator a lowers the eigenstate of N by one unit, and hence we called a the *lowering operator*. The lowering of the state must have a limit. So, there must be the unique ground state such that

$$a\left|0\right\rangle = \left|0\right\rangle,$$

where $\left|0\right\rangle$ denotes the ground state with $N\left|0\right\rangle = 0\left|0\right\rangle$. Since $\left|0\right\rangle$ is a ground state, a cannot lower its state. In the $\left|x\right\rangle$ basis, this equation becomes

$$\left(x+\frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0(x) = 0$$

whose solution is

$$\Rightarrow \psi_0(x) = \left\langle x|0\right\rangle = Ce^{-\left(\frac{m\omega}{2\hbar}\right)x^2},$$

where $C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$ is the normalization factor. We can find the eigenvalues for H or "eigen-energies" from

$$\begin{aligned} a\left|0\right\rangle &= 0 \\ \Rightarrow N\left|0\right\rangle &= a^\dagger a\left|0\right\rangle \\ \Rightarrow H\left|0\right\rangle &= \hbar\omega\left(a^\dagger a+\frac{1}{2}\right)\left|0\right\rangle = \frac{\hbar\omega}{2}\left|0\right\rangle \end{aligned}$$

Therefore, the eigen-energies for $\left|0\right\rangle$ is $\frac{\hbar\omega}{2}$, which is the ground state energy of SHO. Note that the ground state energy, which includes quantum fluctuations, is not zero, as we expect for classical SHO. Now, we can create the higher energy states form the ground state $\left|0\right\rangle$. Consider

$$\begin{aligned} N\left(a^\dagger\left|n\right\rangle\right) &= a^\dagger aa^\dagger\left|n\right\rangle \\ &= a^\dagger\left(a^\dagger a+1\right)\left|n\right\rangle, \quad \text{since } aa^\dagger - a^\dagger a = 1 \\ &= (n+1)\left[a^\dagger\left|n\right\rangle\right]. \end{aligned} \tag{4.2}$$

The state $a^\dagger\left|n\right\rangle$ is the same state as $\left|n+1\right\rangle$. Therefore, the operator $a^\dagger$ is called the *raising operator*. We note that we can find the relationship in Eq. 4.2 and Eq. 4.2 using the commutators

$$\begin{aligned} \left[N,a^\dagger\right] &= a^\dagger \\ \left[N,a\right] &= -a \end{aligned}$$

So now, using $a^\dagger$, we can create a "tower" of states starting from the ground state $\left|0\right\rangle$:

$$\begin{aligned} \left|1\right\rangle &= C_1a^\dagger\left|0\right\rangle \\ \left|2\right\rangle &= C_2a^\dagger\left|1\right\rangle \\ \left|2\right\rangle &= C_3a^\dagger\left|2\right\rangle \\ &\vdots \\ \left|n\right\rangle &= C_na^\dagger\left|n-1\right\rangle \\ &\vdots \end{aligned}$$

where C_n are constants to be determined using the normalization condition, that is, $\langle n|n\rangle = 1$:

$$\langle n+1|n+1\rangle = 1 \Rightarrow |C_{n+1}|^2 \left(\langle n| a\right)\left(a^\dagger |n\rangle\right) = |C_{n+1}|^2 \left\langle n \,\middle|\, aa^\dagger \,\middle|\, n\right\rangle = |C_{n+1}|^2 \left\langle n \,\middle|\, a^\dagger a + 1 \,\middle|\, n\right\rangle = |C_{n+1}|^2(n+1) = 1$$

Therefore,

$$C_{n+1} = \frac{1}{\sqrt{n+1}}$$

and we have

$$|n+1\rangle = \frac{1}{\sqrt{n+1}} a^\dagger |n\rangle \quad \text{or} \quad a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,.$$

Similarly, for the lowering operator a, we have

$$\langle n-1|n-1\rangle = 1 \Rightarrow |C_{n-1}|^2 \left(\langle n| a^\dagger\right)\left(a\,|n\rangle\right) = |C_{n-1}|^2 \left\langle n \,\middle|\, a^\dagger a \,\middle|\, n\right\rangle = |C_{n-1}|^2 n = 1$$

Therefore,

$$C_{n-1} = \frac{1}{\sqrt{n}}$$

and we have

$$|n-1\rangle = \frac{1}{\sqrt{n}} a\,|n\rangle \quad \text{or} \quad a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,.$$

So, given that $|0\rangle$ is normalized, all $|n\rangle$'s generated using $a^\dagger$ will also be normalized. In general, we can create $|n\rangle$ from the ground state by applying $a^\dagger$:

$$|n\rangle = \frac{a^\dagger}{\sqrt{n}}\,|n-1\rangle = \frac{\left(a^\dagger\right)^2}{\sqrt{n\cdot(n-1)}}\,|n-2\rangle = \frac{\left(a^\dagger\right)^3}{\sqrt{n\cdot(n-1)\cdot(n-2)}}\,|n-3\rangle = \cdots = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}\,|0\rangle\,.$$

We also note that $|n\rangle$ form an orthonormal basis, in particularly,

$$\langle n|n'\rangle = \delta_{nn'}.$$

So, the eigenvalue problem for SHO is completely solve with

$$H\,|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle = E_n\,|n\rangle\,,$$

where $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ is an eigenenergy and $|n\rangle$ is the associated eigenstate. Now some of you might ask how we know that there is no other eigenvalue with n being non-integer. So, let $|\alpha\rangle$ be an eigenstate with the associated eigenvalue $E_\alpha = \hbar\omega\left(\alpha + \frac{1}{2}\right)$ with $\alpha \notin \mathbb{Z}$:

$$H\,|\alpha\rangle = E_\alpha\,|\alpha\rangle$$

Now we lower the state $|\alpha\rangle$ using a^k, for some k α will be negative, that is, $\alpha - k \leqslant 0$ but

$$\langle \alpha - k|a^\dagger a|\alpha - k\rangle = \langle \alpha - k|N|\alpha - k\rangle = \alpha - k \geqslant 0,$$

which is a contradiction. Also, as we will see that the energy of SHO cannot be negative since both kinetic and potential terms are always positive. Therefore, negative n or negative energy is not allowed. Therefore, $n \in \mathbb{Z}$ and $a^\dagger$ guarantees that all positive integers are included with $n = 0$ being the unique ground state. Hence:

1. All eigenstates are included.

2. $|n\rangle$ forms a complete basis for the Hilbert space H.
3. Once $|n\rangle$ is chosen as a basis, all operators can be represented in a (countably infinite dimension) matrices using the $|n\rangle$ basis.

For the position operator X, we have

$$\langle n' \,|\, X \,|\, n\rangle = \left\langle n' \left| \sqrt{\frac{\hbar}{2m\omega}} \left(a + a^\dagger\right) \right| n \right\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left[\delta_{n,n'+1}\sqrt{n} + \delta_{n',n+1}\sqrt{n'}\right]$$

In a matrix from, we have

$$X = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Similarly, for the momentum operator, we have

$$\langle n' \,|\, P \,|\, n\rangle = \left\langle n' \left| i\sqrt{\frac{m\hbar\omega}{2}} \left(-a + a^\dagger\right) \right| n \right\rangle = i\sqrt{\frac{m\hbar\omega}{2}} \left[-\delta_{n,n'+1}\sqrt{n} + \delta_{n',n+1}\sqrt{n'}\right]$$

and in a matrix form,

$$P = i\sqrt{\frac{m\hbar\omega}{2}} \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

You can check that $[X, P] = i\hbar\mathbb{1}$; however, note that both X and P are not of finite dimension (do you remember Question 3 in Problem Set 2?) We can write a, $a^\dagger$, and H in the matrix form using the $|n\rangle$ basis; I will leave that as an exercise for you.

Now if we want to write the wave function $\langle x|n\rangle$ for the state $|n\rangle$, we can start from the following relation

$$|n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}} |0\rangle$$

and "project" $|n\rangle$ onto the position basis

$$\phi_n(x) \equiv \langle x|n\rangle = \left\langle x \left| \frac{\left(a^\dagger\right)^n}{\sqrt{n!}} \right| 0 \right\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} \left(\sqrt{\frac{m\omega}{\hbar}}\right)^{n+1/2} \left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n e^{-\frac{m\omega}{2\hbar}x^2}.$$

This function will generate the Hermite polynomial. Let $y = \sqrt{\frac{m\omega}{\hbar}}x$, then we can rewrite the above equation as

$$\begin{aligned} \phi_n(x) &= \langle x|n\rangle = \left\langle x \left| \frac{\left(a^\dagger\right)^n}{(n!)^{1/2}} \right| 0 \right\rangle \\ \Rightarrow \quad \phi_n(y) &= \frac{1}{(2^n n!)^{1/2}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left[\left(y - \frac{d}{dy}\right)^n e^{-y^2/2}\right], \end{aligned}$$

which can be compared with the solution we obtain from the previous section:

$$\phi_n(y) = \frac{1}{(2^n n!)^{1/2}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-y^2/2} H_n(y)$$

By comparing these two equations, we obtain the formulation for the Hermite polynomials, that is,

$$\begin{aligned} e^{-y^2/2} H_n(y) &= \left(y - \frac{d}{dy}\right)^n e^{-y^2/2} \\ \Rightarrow \quad H_n(y) &= e^{y^2/2} \left(y - \frac{d}{dy}\right)^n e^{-y^2/2} \end{aligned}$$

We can use this expression to generate the Hermites polynomials. Note that this expression is consistent with $H_0(y) = 1$ and $H_1(y) = 2y$.

4.2 Hermite Polynomial $H_n(y)$

Some part of this section might be too much mathematical, which might not provide you with any physics insight. However, I still encourage you to go through physics this section and hopefully you can appreciate its nice result at the end. In this section, we will discuss in more detail properties of Hermite polynomials since they play the most important role in the eigen-wavefunction of the harmonic oscillators. From the previous section, we have learned that the eigen-wavefunction $\phi_n(y)$ for the harmonic oscillators can be written as

$$\phi_n(y) = N_n H_n(y) e^{-\frac{1}{2}y^2}, \tag{4.3}$$

where N_n is a normalization factor, $y = \sqrt{\frac{m\omega}{\hbar}}x$, and $H_n(y)$ is the Hermite polynomial of order n. The differential equation that we have solved to obtain the Hermite polynomials is

$$H''(y) - 2yH'(y) + (2\epsilon - 1) H(y) = 0.$$

The Hermite polynomial can be expressed in terms of a generating function $S(y, s)$, where

$$\begin{aligned} S(y,s) &= e^{y^2-(s-y)^2} = e^{-s^2+2sy} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-s^2 + 2sy\right)^n \\ &\equiv \sum_{n=0}^{\infty} \frac{H_n(y)}{n!} s^n \end{aligned} \tag{4.4}$$

In order to obtain the equation above, all we have to do is to match the power of s, and the polynomial in y with the same power n of s will give us the Hermite polynomial of order n. For example,

n	n^{th} term in Eq. 4.4	s^n term	$H_n(y)$
0	1	1	1
1	$-s^2 + 2sy$	$(2y)s$	$2y$
2	$\frac{s^4 - 4s^3y + 4s^2y^2}{2!}$	$\left(2y^2 - 1\right) s^2$	$4y^2 - 2$

Note that a powder n of s is associated only with powers of y that are equal to n or less than n by a multiply of two. Therefore, if all terms of the Hermite polynomial must have either odd or even power but not both.

So, how dow the generating function help us? It turns out that if we can write $S(y,s) = \sum_{n=0}^{\infty} \frac{H_n(y)}{n!} s^n$, then we have

$$\begin{aligned} S(s=0) &= H_0(y) \\ \left.\frac{\partial S}{\partial s}\right|_{s=0} &= H_1(y) \\ \left.\frac{\partial^2 S}{\partial s^2}\right|_{s=0} &= H_2(y) \\ &\vdots \\ \left.\frac{\partial^n S}{\partial s^n}\right|_{s=0} &= H_n(y), \end{aligned}$$

which make it more convenient to calculate the Hermite polynomial. And since we can write

$$S(s,y) = e^{y^2-(s-y)^2},$$

the derivative to order n of $S(s,y)$ can be readily calculated, that is,

$$\frac{\partial^n S}{\partial s^n} = e^{y^2} \frac{\partial^2}{\partial s^n} e^{-(s-y)^2} = (-1)^n e^{y^2} \frac{\partial^n}{\partial y^n} e^{-(s-y)^2}$$

Therefore, we have

$$H_n(y) = \left.\frac{\partial^n S}{\partial s^n}\right|_{s=0} = (-1)^n e^{y^2} \frac{\partial^n}{\partial y^n} e^{-y^2},$$

which is in fact equivalent to another definition of the Hermite polynomial that we have derived in class, that is,

$$H_n(y) = (-1)^n e^{y^2/2} \left(y - \frac{\partial}{\partial y} \right)^n e^{-y^2/2}$$

So now we know how to generate $H_n(y)$ using two different generating functions, but to go back to our question, how does it help us? The usefulness of the generating function becomes clear when we have to deal with an integral involving the Hermite polynomials. For example, if we want to calculate the normalization factor N_n in Eq. 4.3. We will rewrite Eq. 4.3 in terms x

$$\phi_n(x) = N_n H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2},$$

where $\alpha = \sqrt{\frac{m\omega}{\hbar}}$. N_n can be calculated from

$$\langle \phi_n | \phi_n \rangle = \int_{-\infty}^{\infty} |\phi_n(x)|^2 \; dx = \frac{|N_n|^2}{\alpha} \int_{-\infty}^{\infty} H_n^2(y) e^{-y^2} = 1,$$

where again we change the variable back to $y = \alpha x$. The integral that we want to calculate here is $\int_{-\infty}^{\infty} H_n^2(y) e^{-y^2}$. In order to evaluate this integral we write the Hermite polynomial in terms of the generating function $S(y,s)$. Since we have

$$e^{-s^2+2sy} = \sum_{n=0}^{\infty} \frac{H_n(y)}{n!} s^n,$$

we can write

$$\int_{-\infty}^{\infty} e^{-s^2+2sy} \cdot e^{-t^2+2ty} \cdot e^{-y^2}\ dy = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{s^k t^l}{k!l!} \int_{-\infty}^{\infty} H_k(y)H_l(y)e^{-y^2}\ dy$$

The integral on the left hand side can be rewritten as an integral of a Gaussian.

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-s^2+2sy} \cdot e^{-t^2+2ty} \cdot e^{-y^2}\ dy &= \int_{-\infty}^{\infty} e^{-\left(y^2-2y(s+t)+\left(s^2+2st+t^2\right)\right)} e^{2st}\ dy \\ &= e^{2st}\int_{-\infty}^{\infty} e^{-y-(s+t)^2}\ dy \\ &= \sqrt{\pi}e^{2st},\end{aligned}$$

which can be written as a power series as

$$\sqrt{\pi}e^{2st} = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{(2st)^n}{n!} \tag{4.5}$$

For the right hand side, we will rewrite it as

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{s^k t^l}{k!l!}\int_{-\infty}^{\infty} H_k(y)H_l(y)e^{-y^2}\ dy = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{s^k t^l}{k!l!}C_{kl}, \tag{4.6}$$

that is,

$$C_{kl} = \int_{-\infty}^{\infty} H_k(y)H_l(y)e^{-y^2}\ dy$$

Comparing Eqs. 4.5 and 4.6, we can see that if k and l are not equal to n, C_{kl} must be zero, which proves the orthogonality of the Hermite polynomials. If k and l are equal to n, then we have

$$\begin{aligned}\sqrt{\pi}\sum_{n=0}^{\infty}\frac{2^n s^n t^n}{n!} &= \sum_{n=0}^{\infty}\frac{s^n t^n}{n!n!}C_{nn} \\ \Rightarrow\quad C_{nn} &= \sqrt{\pi}2^n n!.\end{aligned}$$

Therefore, we have

$$\int_{-\infty}^{\infty} H_n(y)H_m(y)e^{-y^2}\ dy = \sqrt{\pi}2^n n!\delta_{nm}.$$

Now back to finding the normalization factor,

$$\begin{aligned}\frac{|N_n|^2}{\alpha}\int_{-\infty}^{\infty} H_n(y)H_m(y)e^{-y^2}\ dy &= \frac{|N_n|^2}{\alpha}\sqrt{\pi}2^n n! = 1 \\ \Rightarrow\quad N_n &= \left(\frac{\alpha}{\sqrt{\pi}2^n n!}\right)^{1/2}\end{aligned}$$

4.3 Time evolution of Gaussian wave function

Now we will move on to a more-physics-related topic, and we will apply what we have learned about the Hermite polynomial to help us with the next problem. The problem is a Gaussian wave packet initially centered at $x = a$ in the harmonic potential. The initial wave function can be described by

$$\psi(x,0) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-\frac{1}{2}\alpha^2(x-a)^2}.$$

Note that if $a = 0$, then $\psi(x, 0)$ is the $n = 0$ eigenstate of the harmonic potential, and it would have been a stead state, which is not very interesting. As you will see shortly that it is a lot of interesting when $a \neq 0$. The question we want to answer here is, what does the time evolution of $\psi(x, 0)$ look like? In order to write down $\psi(x, t)$, we will follow the procedure that we all are familiar now, that is, we have to first expand $\psi(x, 0)$ in terms of the eigen-wavefunctions of the harmonic oscillators.

$$\psi(x,0) = \sum_{n=0}^{\infty} A_n \phi_n(x),$$

where

$$\begin{aligned}
A_n &= \langle \phi_n | \psi \rangle = \int_{-\infty}^{\infty} dx \, \langle \phi_n | x \rangle \langle x | \psi \rangle \\
&= \int_{-\infty}^{\infty} \phi_n^*(x) \psi(x,0) \, dx \\
&= \left(\frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} \left(\frac{\alpha^2}{\pi} \right)^{1/4} \frac{1}{\alpha} \int_{-\infty}^{\infty} H_n(y) e^{-y^2/2} \cdot e^{-\frac{1}{2}(y-y_0)^2} \, dy \\
&= \left(\frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} \left(\frac{\alpha^2}{\pi} \right)^{1/4} \frac{1}{\alpha} \int_{-\infty}^{\infty} H_n(y) e^{-\left(y^2 - y y_0 + \frac{1}{2} y_0^2\right)} \, dy,
\end{aligned}$$

where $y_0 = \alpha a$. Next we apply the same trick that we have see to calculate the normalization factor to evaluate this integral for A_n.

$$\int_{-\infty}^{\infty} e^{-s^2 + 2sy} \cdot e^{-\left(y^2 - y y_0 + \frac{1}{2} y_0^2\right)} \, dy = \sum_{n=0}^{\infty} \frac{s^n}{n!} \int_{-\infty}^{\infty} H_n(y) e^{-\left(y^2 - y y_0 + \frac{1}{2} y_0^2\right)} \, dy \tag{4.7}$$

After performing the Gaussian integral that by now we are familiar with, the left hand side becomes

$$\sqrt{\pi} e^{-\frac{1}{4} y_0^2 + s y_0} = \sqrt{\pi} e^{-\frac{1}{4} y_0^2} \sum_{n=0}^{\infty} \frac{s^n y_0^n}{n!}$$

Equating the left hand side and the right hand side of Eq. 4.7, we obtain

$$\int_{-\infty}^{\infty} H_n(y) e^{-\left(y^2 - y y_0 + \frac{1}{2} y_0^2\right)} \, dy = \sqrt{\pi} e^{-\frac{1}{4} y_0^2} y_0^n.$$

Therefore,

$$A_n = \left(\frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} \left(\frac{\alpha^2}{\pi} \right)^{1/4} \frac{1}{\alpha} \sqrt{\pi} e^{-\frac{1}{4} y_0^2} y_0^n = \frac{y_0^n e^{-\frac{1}{4} y_0^2}}{(2^n n!)^{1/2}}$$

And in terms of the eigen-wavefunctions, we can write $\psi(x, 0)$ as

$$\psi(x,0) = \sum_{n=0}^{\infty} \frac{y_0^n e^{-\frac{1}{4} y_0^2}}{(2^n n!)^{1/2}} \left(\frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} H_n(\alpha x) e^{-\frac{1}{2} \alpha^2 x^2}$$

The time evolution of this state can be easily written as (note that we interchange x back to y again)

$$\psi(y,t) = \sum_{n=0}^{\infty} \frac{y_0^n e^{-\frac{1}{4} y_0^2}}{(2^n n!)^{1/2}} \left(\frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} H_n(y) e^{-\frac{1}{2} y^2} e^{-i\omega t(n+1/2)},$$

which we can rearrange all those terms and rewrite as

$$\psi(x,t) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-\frac{1}{2}y^2-\frac{1}{4}y_0^2-\frac{1}{2}i\omega t} \sum_{n=0}^{\infty} \frac{H_n(y)}{n!} \left(\frac{1}{2}y_0 e^{-i\omega t}\right)^n .$$

The last term looks the same as the generating function if we let $s = \frac{1}{2}y_0 e^{-i\omega t}$, and hence we have

$$\psi(x,t) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-\frac{1}{2}y^2-\frac{1}{4}y_0^2-\frac{1}{2}i\omega t} \cdot \exp\left(-\frac{1}{4}y_0^2 e^{-2i\omega t} + yy_0 e^{-i\omega t}\right)$$

Now it is a matter of rearranging term

$$\begin{aligned}
\psi(x,t) &= \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-\frac{1}{2}y^2-\frac{1}{4}y_0^2-\frac{1}{2}i\omega t} \cdot \exp\left(-\frac{1}{4}y_0^2 e^{-2i\omega t} + yy_0 e^{-i\omega t}\right) \\
&= \left(\frac{\alpha^2}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}y^2 - \frac{1}{4}y_0^2 - \frac{1}{2}i\omega t - \frac{1}{4}y_0^2\left(\cos 2\omega t - i\sin 2\omega t\right) + yy_0\left(\cos\omega t - i\sin\omega t\right)\right] \\
&= \left(\frac{\alpha^2}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}y^2 - \frac{1}{4}y_0^2 + \frac{1}{4}y_0^2 - \frac{1}{2}y_0^2\cos^2\omega t + yy_0\cos\omega t + i\left(-\frac{1}{2}\omega t + \frac{1}{4}\sin 2\omega t - yy_0\sin\omega t\right)\right] \\
&= \left(\frac{\alpha^2}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}y^2 - \frac{1}{2}y_0^2\cos^2\omega t + yy_0\cos\omega t + i\phi(y,t)\right] \\
&= \left(\frac{\alpha^2}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}\left(y - y_0\cos\omega t\right)^2 - i\phi(y,t)\right] \\
\psi(x,t) &= \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-\frac{1}{2}(y-y_0\cos\omega t)^2} \cdot e^{i\phi(y,t)}
\end{aligned}$$

The probability of finding the particle at later time t is, therefore, equal to

$$P = |\psi(x,t)|^2 = \frac{\alpha}{\pi^{1/2}} e^{-\alpha^2(x-a\cos\omega t)^2},$$

which shows that the center of the Gaussian $a\cos\omega t$, where $a = y_0/\alpha$, moves as if it is oscillating with the frequency ω inside the harmonic potential (also see Fig. 4.1). This behavior is what we expect from a classical object, which in this case can be represented by a Gaussian, in the harmonics potential.

4.4 Coherent States

In an attempt to create a quantum state that resembles a classical particle in the harmonic oscillator, in this section we will discuss the coherent state. However, we will state a question that at first seems to be irrelevant to the task we want to accomplish. That question is, what is an eigenstate of the lowering operator?

In order to simplify the problem of finding eigenstates and eigen-energies in the harmonic potential, we have introduced the raising and lower operators, which are defined as

$$\begin{aligned}
a &= \frac{X}{x_0} + i\frac{P}{p_0} \\
a^\dagger &= \frac{X}{x_0} - i\frac{P}{p_0},
\end{aligned}$$

where $x_0 = \sqrt{\frac{2\hbar}{m\omega}}$ and $p_0 = \sqrt{2\hbar m\omega}$. The eigenvalue problem that we have to solve for the lowering operator a is

$$a\left|\lambda\right\rangle = \lambda\left|\lambda\right\rangle,$$

where λ is an eigenvalue and $\left|\lambda\right\rangle$ is an eigenstate. Note that a is not Hermitian, that is, $a \neq a^\dagger$. Therefore, λ can be a complex number.

We will start by noticing some similarity between $\left[a, a^\dagger\right]$ and $[X, P]$; both can be described by

$$\begin{aligned}\left[a, a^\dagger\right] &= 1 \\ \left[P, \frac{i}{\hbar}X\right] &= 1,\end{aligned}$$

and since you know that we can write

$$P \rightarrow \frac{\hbar}{i}\frac{d}{dx},$$

by the same analogy, we must be able to write

$$a \rightarrow \frac{d}{da^\dagger}.$$

In addition, we will assume that we can write the eigenstate $\left|\lambda\right\rangle$ as

$$\left|\lambda\right\rangle = \psi(\lambda)\left|0\right\rangle.$$

You might ask why we think writing $\left|\lambda\right\rangle$ this way would work. The best answer I can give here is that because we already know that we can create all eigenstates $\left|n\right\rangle$ of the harmonic oscillators from the ground state $\left|0\right\rangle$ using $a^\dagger$ and we can write any state in particular $\left|\lambda\right\rangle$ in terms of superposition of $\left|n\right\rangle$. Therefore, the eigenvalue problem becomes

$$\begin{aligned}\frac{d}{da^\dagger}\psi(\lambda)\left|0\right\rangle &= \lambda\psi(\lambda)\left|0\right\rangle \\ \Rightarrow \quad \frac{d}{da^\dagger}\psi(\lambda) &= \lambda\psi(\lambda) \\ \Rightarrow \quad \psi(\lambda) &= C_\lambda e^{\lambda a^\dagger}\end{aligned}$$

We can check that $|\lambda\rangle = C_\lambda e^{\lambda a^\dagger}|0\rangle$ is the eigenstate of a:

$$\begin{aligned}
a|\lambda\rangle &= aC_\lambda e^{\lambda a^\dagger}|0\rangle \\
&= aC_\lambda \sum_{n=0}^{\infty} \frac{\left(\lambda a^\dagger\right)^n}{n!}|0\rangle \\
&= C_\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} a\left[\frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0\rangle\right] \\
&= C_\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} a|n\rangle \\
&= C_\lambda \sum_{n=1}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \sqrt{n}|n-1\rangle \\
&= C_\lambda \sum_{n=1}^{\infty} \lambda \frac{\lambda^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle \\
&= \lambda C_\lambda \sum_{n'=0}^{\infty} \frac{\lambda^{n'}}{\sqrt{n'!}}|n'\rangle, \quad \text{where } n' = n-1 \\
&= \lambda|\lambda\rangle
\end{aligned}$$

We can also calculate the normalization factor C_λ. We want to calculate C_λ such that

$$\langle\lambda|\lambda\rangle = 1.$$

Substituting $|\lambda\rangle$ we obtained above, we have

$$\begin{aligned}
\langle\lambda|\lambda\rangle &= \langle 0|C_\lambda^*\left(e^{\lambda a^\dagger}\right)^\dagger C_\lambda e^{\lambda a^\dagger}|0\rangle \\
&= |C_\lambda|^2 \langle 0|e^{\lambda^* a} e^{\lambda a^\dagger}|0\rangle \\
&= |C_\lambda|^2 \sum_{n=0}^{\infty} \frac{(\lambda^*)^n}{n!} \langle 0|a^n e^{\lambda a^\dagger}|0\rangle,
\end{aligned}$$

but we know that

$$ae^{\lambda a^\dagger}|0\rangle = \lambda e^{\lambda a^\dagger}|0\rangle$$

and hence

$$a^n e^{\lambda a^\dagger}|0\rangle = \lambda^n e^{\lambda a^\dagger}|0\rangle.$$

Therefore,

$$\begin{aligned}
\langle\lambda|\lambda\rangle &= |C_\lambda|^2 \sum_{n=0}^{\infty} \frac{(\lambda^*)^n}{n!} \lambda^n \langle 0|e^{\lambda a^\dagger}|0\rangle \\
&= |C_\lambda|^2 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} \langle 0|e^{\lambda a^\dagger}|0\rangle
\end{aligned}$$

So, what is $\langle 0|e^{\lambda a^\dagger}|0\rangle$?

$$\begin{aligned}
\langle 0|e^{\lambda a^\dagger}|0\rangle &= \langle 0| \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left(a^\dagger\right)^m |0\rangle \\
&= \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \langle 0| \left(a^\dagger\right)^m |0\rangle,
\end{aligned}$$

but $\langle 0| \left(a^\dagger\right)^m |0\rangle = 0$ unless $m = 0$. Therefore,

$$\langle 0|e^{\lambda a^\dagger}|0\rangle = 1.$$

And hence we have

$$\begin{aligned} \langle\lambda|\lambda\rangle &= |C_\lambda|^2 \sum_{n=0}^{\infty} \frac{\left(|\lambda|^2\right)^n}{n!} \\ &= |C_\lambda|^2 \cdot e^{|\lambda|^2} \\ \Rightarrow \quad C_\lambda &= e^{-|\lambda|^2/2}. \end{aligned}$$

Therefore, the normalized eigenstate of a is

$$\boxed{|\lambda\rangle = e^{-|\lambda|^2/2} \cdot e^{\lambda a^\dagger}\,|0\rangle}$$

As you can imagine, it is difficult to visualize this state when it is written in this form. Therefore, in order to understand this state better, we will try to write it in the coordinate basis. Hopefully, we will obtain more insight into this state. So next we want to calculate

$$\psi_\lambda(x) = \langle x|\lambda\rangle$$

From the eigenvalue equation, we know that

$$\begin{aligned} a\,|\lambda\rangle &= \lambda\,|\lambda\rangle \\ \Rightarrow \quad \langle x|a|\lambda\rangle &= \lambda\,\langle x|\lambda\rangle = \lambda\psi_\lambda(x). \end{aligned}$$

In the x-basis, the lowering operator a becomes

$$a = \frac{X}{x_0} + i\frac{P}{p_0} \to \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right).$$

The equation for the wave function then becomes

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_\lambda(x) = \lambda\psi_\lambda(x).$$

Let $d^2 = \dfrac{\hbar}{m\omega}$, then

$$\begin{aligned} \left(x + d^2\frac{d}{dx}\right)\psi_\lambda(x) &= \sqrt{2}d\lambda\psi_\lambda(x) \\ \int d^2\frac{d\psi_\lambda}{\psi_\lambda} &= \int\left(\sqrt{2}d\lambda - x\right)dx \\ d^2\log\psi_\lambda(x) &= \sqrt{2}d\lambda x - \frac{1}{2}x^2 + C_0 \\ &= -\frac{1}{2}\left(x - \sqrt{2}d\lambda\right)^2 + \left(d^2\lambda^2 + C_0\right) \\ \Rightarrow \quad \psi_\lambda(x) &= C_1 e^{-\frac{1}{2d^2}\left(x-\sqrt{2}d\lambda\right)^2}, \end{aligned} \tag{4.8}$$

where $C_1 = e^{\lambda^2 + C_0/d^2}$ is a normalization factor, and hence it is equal to $\left(\dfrac{1}{\pi d^2}\right)^{1/4}$. If we let $x_0 = \sqrt{2}d\lambda$ and $d = 1/\alpha$, then it is clear that the coherent state in the coordinate basis is in fact a Gaussian wave packet which centers at $x = x_0$, that is,

$$\boxed{\psi_\lambda(x) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-\frac{1}{2}\alpha^2(x-x_0)^2}}$$

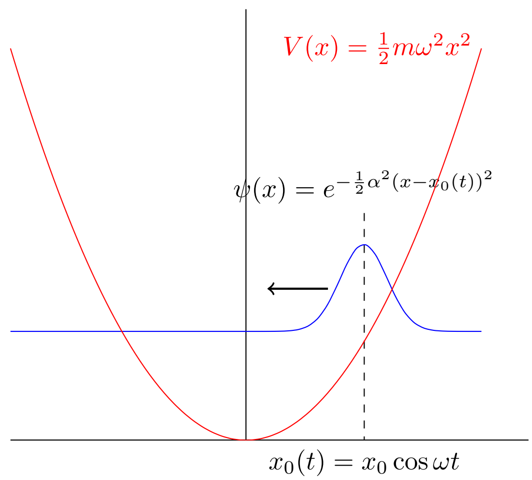

Figure 4.1: The coherent state of the one dimensional harmonic oscillators in the coordinate basis is a Gaussian centered at $x = x_0$

The coherent state resembles a classical particle located represented by a Gaussian centered at x_0 (Fig. 4.1), and it is one of the closest quantum mechanical state that is analogous or can be related to a classical state of a particle. In the next section, we will discuss the time evolution of this Gaussian wave packet in the harmonic potential, and show the oscillation of the coherent state.

How if we substitute back $d = \sqrt{\frac{\hbar}{m\omega}}$, then our wave function of the coherent state will become

$$\psi_\lambda(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}(x-x_0)^2},$$

which looks very similar to the eigen-wavefunction of the ground state,

$$\langle x|0\rangle \equiv \phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}.$$

The only difference is that the coherent state is translated to be centered at $x = x_0$ instead of at $x = 0$. Therefore we can view the coherent state as the translated ground state to $x = x_0 = \sqrt{2}d\lambda$, that is,

$$|\lambda\rangle = T_{\sqrt{2}d\lambda}|0\rangle,$$

where $T_{\sqrt{2}d\lambda}$ is the translation operator of a distance $\sqrt{2}d\lambda$. We will next learn how to define the coherent state using the translation operator and try to generalize our formulation.

First, let us consider properties of the translation operator. We have discussed the translation operator before when we introduced the momentum operator as the generator of the translation. In that case, we only consider an infinitesimal translation. However, now we are interested in a finite translation, which is no small. The translation operator can be defined as

$$T_{x_0} = e^{-\frac{i}{\hbar}x_0 P},$$

where x_0 is the amount of the translation and P is the momentum operator. We can show that

$$T_{x_1}T_{x_2} = e^{-\frac{i}{\hbar}x_0 P} \cdot e^{-\frac{i}{\hbar}x_0 P} = e^{-\frac{i}{\hbar}(x_1+x_2)P} = T_{x_1+x_2},$$

and

$$T_{x_0}^\dagger = e^{\frac{i}{\hbar}x_0 P} = e^{-\frac{i}{\hbar}(-x_0)P} = T_{-x_0} = T_{x_0}^{-1},$$

which confirms that the translation operator is a unitary operator.

The transformation of the position operator X under the translation is given by

$$T_{x_0}^\dagger X T_{x_0} = e^{\frac{i}{\hbar}x_0 P} X e^{-\frac{i}{\hbar}x_0 P}.$$

Now, we can use the following identity to simplify this expression.

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots = \sum_0^\infty \frac{1}{n!}A^n\{B\},$$

where

$$A^0\{B\} = B,\ A^1\{B\} = [A,B],\ A^2\{B\} = [A,[A,B]],\ \cdots.$$

From the above expression, we will let $A = \frac{i}{\hbar}x_0 P$ and $B = X$, and hence $[\frac{i}{\hbar}x_0 P, X] = \frac{i}{\hbar}x_0(-i\hbar) = x_0$. Therefore, we have

$$T_{x_0}^\dagger X T_{x_0} = e^{\frac{i}{\hbar}x_0 P} X e^{-\frac{i}{\hbar}x_0 P} = X + [\frac{i}{\hbar}x_0 P, X] = X + x_0.$$

Note that all higher order terms are equal to zero because x_0 is a scalar and hence it commutes with an operator. From this result, we can see that the transformation of the position operator under the translation is what we expected, that is, the position operator is translated by x_0 along the positive direction.

Next, we will consider a case where the translation acts on the x-basis and an arbitrary state. We expect that when the translation acts on the x-basis, it will translate x by the amount of the translation, hence when T_{x_0} acts $|x\rangle$, we have

$$T_{x_0}|x\rangle = |x + x_0\rangle.$$

In order to consider a case, when T_{x_0} acts on $\langle x|$, let us consider

$$T_{x_0}^\dagger|x\rangle = T_{x_0}^{-1}|x\rangle = T_{-x_0}|x\rangle = |x - x_0\rangle.$$

Now, since in changing from $|\rangle$ to $\langle|$, we have

$$T_{x_0}^\dagger|x\rangle \longrightarrow \langle x| T_{x_0}.$$

Therefore,

$$\langle x| T_{x_0} = \langle x - x_0|.$$

Now we are ready to consider the case when T_{x_0} acts on an arbitrary state, and its wave function.

$$\langle x|T_{x_0}|\psi\rangle = \langle x - x_0|\psi\rangle = \psi(x - x_0).$$

We can see that when we apply the translation to the state $|\psi\rangle$, the wave function $\psi(x)$ gets translated along the positive x-axis by the amount x_0 as we expected.

After defining the translation operator, we are now ready to define the coherent state using the translation operator.

$$\boxed{|\lambda\rangle = T_{x_0}|0\rangle = e^{-\frac{i}{\hbar}x_0 P}|0\rangle},$$

where $x_0 = \sqrt{2}d\lambda = \sqrt{\frac{2\hbar}{m\omega}}\lambda$ or $\lambda = \frac{x_0}{\sqrt{2}d}$. Therefore, for comparison, from our previous discussion, we can rewrite the coherent state in terms of d and x_0 as

$$|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle = e^{-\frac{x_0^2}{4d^2}}e^{\frac{x_0}{\sqrt{2}d}a^\dagger}|0\rangle$$

In the homework this week, you will be asked to show that these two forms of the coherent state are in fact equivalent.