Lecture 6: Rotation Groups and Angular Momentum

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In this lecture, we will formulate angular momentum operators in quantum mechanics. One of the conventional method that you probably have learned in Quantum Mechanics I is to start from the definition of the angular momentum in classical mechanics, which is $\vec{L} = \vec{r} \times \vec{P}$, and then write the angular momentum operators $\hat{L}$ in term of the $\hat{X}$ and $\hat{P}$ operators. However, in this lecture, we will try to formulate $\hat{L}$ by relying on a rotation operator and define $\hat{L}$ as a generator of the rotation. Therefore, we will start the discussion of the angular momentum operators in quantum mechanics by pointing out similarities and differences between two rotation groups $SO(3)$ and $SU(2)$. The former is used to describe the rotation of vectors in classical three-dimensional space, whereas the latter defines the rotation of states and operators in quantum mechanics.

6.1 Rotation groups

We begin by stating the definition of a group.

Definition 6.1 *A group G has the following properties:*

G1: If g and h are in G then $g \cdot h$ is also in G. (closed)

G2: There exists an identity 1 such that $1 \cdot g = g \cdot 1 = g$. (identity)

G3: There exists g^{-1} such that $g^{-1}g = g^{-1} = 1$. (inverse)

G4: $f \cdot (g \cdot h) = (f \cdot g) \cdot h$. (associative)

So, from this definition, we can see that rotations form a group. And, what is the rotation group? One of the natural candidates is the rotation group in $\mathbb{R}^3$, which is called a special orthogonal group of $\mathbb{R}^3$ or $SO(3)$. The properties of $SO(3)$ are:

1. Its presentation is real 3×3 special orthogonal matrices.
2. It preserves the inner product, that is, $R^T R = \mathbb{1}$. This property is denoted by "O" for orthogonal in $SO(3)$.
3. It preserves the orientation, that is, $\det R = 1$. This property is denoted by "S" for special in $SO(3)$.

Examples of the matrix representations of the rotation in $\mathbb{R}^3$ are

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},$$

which represents the rotation around the $x-$axis,

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which represents the rotation around the $z-$axis, and

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

which represents the rotation around the $y-$axis.

We note that we can get to any $\hat{n}$ vector in $\mathbb{R}^3$ by only two rotations. For example, to obtain a unit vector $\hat{n}$, which makes an angle ϕ with the $z-$axis and whose projection on the $xy-$plane makes an angle θ with the x-faxis, first we need to rotate around the $y-$axis by ϕ and then rotate around the $z-$axis by θ.

The rotations around different axes do not commute. For example,

$$R_x(\theta)R_z(\theta) \neq R_z(\theta)R_x(\theta).$$

In other words, the rotation group forms a *non-abelian* group. We will see later that this property is equivalent to the fact that the commutation of two different rotation operators in quantum mechanics is non-zero.

In order to examine the non-commutative nature of rotations, let us consider an infinitesimal rotation, where the angle of rotation ϵ is very small. In this case, the rotations around the $x-$, $y-$ and $z-$ axes become

$$\begin{aligned} R_x(\theta) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1-\frac{\epsilon^2}{2} \end{bmatrix}, \\ R_z(\theta) &= \begin{bmatrix} 1-\frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1-\frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ R_y(\theta) &= \begin{bmatrix} 1-\frac{\epsilon^2}{1} & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\frac{\epsilon^2}{2} \end{bmatrix}. \end{aligned}$$

Note that we only keep up to terms with order ϵ^2. Consider the difference between $R_x(\epsilon)R_y(\epsilon)$ and $R_y(\epsilon)R_x(\epsilon)$

$$R_x(\epsilon)R_y(\epsilon) - R_y(\epsilon)R_x(\epsilon) = \begin{bmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R_z(\epsilon^2) - \mathbb{1} = R_z(\epsilon^2) - R_i(0),$$

where $R_i(0)$ denotes any of the rotation matrices.

6.2 Rotations in quantum mechanics

Having considered the rotations in $\mathbb{R}^3$, next we will describe rotations in quantum mechanics. So, we want to create an operator $\mathcal{D}(R)$ in quantum mechanics to associate with a rotation operator R in classical mechanics. An action of $\mathcal{D}(R)$ is to rotate a state, that is,

$$|\alpha\rangle_R = \mathcal{D}(R)\,|\alpha\rangle\,.$$

The matrix representation of $\mathcal{D}(R)$ depends on the dimensionality N of the Hilbert space that $|\alpha\rangle$ belongs to.

For $N = 2$, the rotation group is $SU(2)$, a special unitary group, whose elements are complex numbers. It preserves norm and orientation, that is, $U^\dagger U = \mathbb{1}$ and $\det U = 1$, respectively. To construct $\mathcal{D}(R)$, we consider the infinitesimal transformation U_ϵ, where

$$U_\epsilon = \mathbb{1} - iG\epsilon,$$

where G is a generator of the transformation. For rotation, we have

$$G \to \frac{J_{\hat{n}}}{\hbar}, \qquad \epsilon \to d\phi.$$

where $J_{\hat{n}} = \vec{J} \cdot \hat{n}$ is the generator of rotation around a unit vector $\hat{n}$. $J_{\hat{n}}$ is in fact the *angular momentum operator*. Therefore, the rotation operator for an infinitesimal case becomes

$$\mathcal{D}(\hat{n}, d\phi) = \mathbb{1} - i\left(\frac{\vec{J}\cdot\hat{n}}{\hbar}\right) d\phi,$$

and for a finite rotation, we have

$$\mathcal{D}(\hat{n}, \phi) = \lim_{N\to\infty}\left[\mathbb{1} - i\left(\frac{\vec{J}\cdot\hat{n}}{\hbar}\right)\frac{\phi}{N}\right]^N = e^{-i\vec{J}\hat{n}\phi/\hbar}.$$

We note that $\vec{J}$ is defined as the generator of rotation and we make no reference to angular momentum in classical mechanics, which is defined as $\vec{L} = \vec{r} \times \vec{p}$. Therefore, this derivation can be applied to both angular momentum operators and spin operators, which have no classical counterpart.

It turns out that R and $\mathcal{D}(R)$ have the same group properties, but their representations are different. As we have seen, the representation of R in $SO(3)$ is a real 3×3 matrix, but the representation of $\mathcal{D}(R)$ for $N = 2$ in $SU(2)$ is a complex 2×2 matrix.

From the mapping $R \to \mathcal{D}(R)$, the infinitesimal rotation in quantum mechanics must satisfy

$$\mathcal{D}(R_x,\epsilon)\mathcal{D}(R_y,\epsilon) - \mathcal{D}(R_y,\epsilon)\mathcal{D}(R_x,\epsilon) = \mathcal{D}(R_z,\epsilon^2) - \mathcal{D}(R_i,0),$$

which can be written as

$$\Rightarrow \left(\mathbb{1} - \frac{iJ_x\epsilon}{\hbar} - \frac{J_x^2\epsilon^2}{2\hbar^2}\right)\left(\mathbb{1} - \frac{iJ_y\epsilon}{\hbar} - \frac{J_y^2\epsilon^2}{2\hbar^2}\right) - \left(\mathbb{1} - \frac{iJ_y\epsilon}{\hbar} - \frac{J_y^2\epsilon^2}{2\hbar^2}\right)\left(\mathbb{1} - \frac{iJ_x\epsilon}{\hbar} - \frac{J_x^2\epsilon^2}{2\hbar^2}\right) = \mathbb{1} - \frac{iJ_z\epsilon^2}{\hbar} - \mathbb{1}.$$

We keep only the ϵ^2 term

$$\begin{aligned} \Rightarrow -\left(J_xJ_y - J_yJ_x\right)\frac{\epsilon^2}{\hbar^2} &= \frac{-iJ_z\epsilon^2}{\hbar} \\ \Rightarrow [J_x, J_y] &= i\hbar J_z. \end{aligned}$$

In general,

$$\boxed{[J_i, J_j] = i\hbar\epsilon_{ijk}J_k}$$

We note that the derivation is based on

1. $J_{\hat{n}}$ is the generator of rotation.
2. Rotations form a non-abelian group, which implies that the rotations around different axes do not commute, which is different from the momentum operator where P_i commutes with one another for different $i = x$, y, and z.

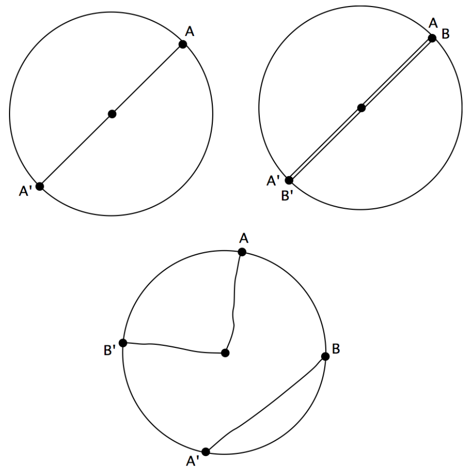


Figure 6.1: Topology of the rotation group $SU(2)$.

6.3 $SO(3)$ vs. $SU(2)$

We have seen that the rotation group representation for spin-1/2 and for three dimensional space are different. The rotation group for spin-1/2 is $SU(2)$ whereas that for three dimensional space is $SO(3)$. One question we can ask is whether or not $SU(2)$ and $SO(3)$ (we note that both groups are the rotation groups) are the same. In order to answer this question, let us consider the rotation by 2π in $SU(2)$ and $SO(3)$. In $SO(3)$ a vector is back to the original state after 2π rotation. However, in $SU(2)$ a vector acquires a non-trivial phase of -1 under 2π rotation. In particular, the rotation operator of spin-1/2 system is given by

$$U(\theta\hat{n}) = e^{-i\vec{S}\cdot\hat{n}\theta/\hbar} = e^{-i\vec{\sigma}\cdot\hat{n}\theta/2} = \cos\frac{\theta}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\theta}{2},$$

and if $\theta = 2\pi$, then

$$U(2\pi) = -1.$$

On the other hand, for $SO(3)$, $U(2\pi) = 1$. Therefore, 2π rotations in $SU(2)$ and $SO(3)$ are different. To illustrate this difference, let us consider a topology of the rotation in $SO(3)$ as a solid sphere in $\mathbb{R}^3$ of radius π, and hence the rotation can be represented by a vector where

1. a unit vector $\hat{n}$ from the center of the sphere represents a direction of the rotation axis.
2. an angle θ is an amount of the rotation around that axis.

The picture of 2π rotation around $\hat{n}$ is shown in the top left figure, represented by a line segment AA' going through the origin. This path is not contractible back to a sphere, and hence is not topologically the same as the case where there is no rotation, which is represented by a sphere without a line. Therefore, topologically, the rotation by 2π is not the same as no rotation. However, if we do the rotation by another 2π represented by a line segment BB' as shown by the top right figure. Now we have two paths that are contractible to a point. By requiring that A and A' are on the opposite side (so are B and B'), we can move points A and B' together and A' and B together. Since both of these lines are contractible to a point, topologically this representation of 4π rotation is the same as no rotation. Therefore, 4π rotation gives back the origin. Therefore, $SO(3)$ is not large enough to describe rotation of half-odd integer spin, *e.g* spin-1/2. We need a larger group, which is $SU(2)$.

Suppose that U is in $SU(2)$ (special unitary group), then $U^\dagger U = \mathbb{1}$ and $\det U = 1$, and we must have

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$, which is an equation for a unit circle. We can show that when written in this form, U satisfies

$$U^\dagger U = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The topology of $SU(2)$ is S^3, which is a sphere in $\mathbb{R}^3$, which is in contrast to a solid sphere for $SO(3)$. Elements of $SU(2)$ group only live on the surface of the sphere and all loops on surface of S^3 are contractible unlike those in $SO(3)$.

We can create a surjective (one-to-one) map from $SU(2)$ to $SO(3)$ using group homomorphism. We note that $SU(2)$ is larger than $SO(3)$. Given $U \in SU(2)$, there is $R(U) \in SO(3)$ such as

$$R(U(a,b)) = \left(\begin{array}{ccc} \Re(a^2 - b^2) & \Im(a^2 + b^2) & -2\Re(ab) \\ -\Im(a^2 - b^2) & \Re(a^2 + b^2) & 2\Im(ab) \\ 2\Re(a\bar{b}) & 2\Im(a\bar{b}) & |a|^2 - |b|^2 \end{array} \right),$$

where $\Re(x)$ and $\Im(x)$ denote the real and imaginary part of $x \in \mathbb{C}$, respectively.

Using this mapping, we can verify that

$$\begin{aligned}
R_x(\theta) &= R\left(U\left(\cos\frac{\theta}{2}, -i\sin\frac{\theta}{2}\right)\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \\
R_y(\theta) &= R\left(U\left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)\right) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \\
R_z(\theta) &= R\left(U\left(e^{-i\theta/2}, 0\right)\right) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

In particular, if $\theta = 0$ and 2π, then we have

$$\begin{aligned}
R_i(0) &= R(U(1,0)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
R_i(2\pi) &= R(U(-1,0)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

where i can be x, y, or z. That is, the two different elements in $SU(2)$ representing the zero and 2π rotations are mapped to the same group element in $SO(3)$. In general,

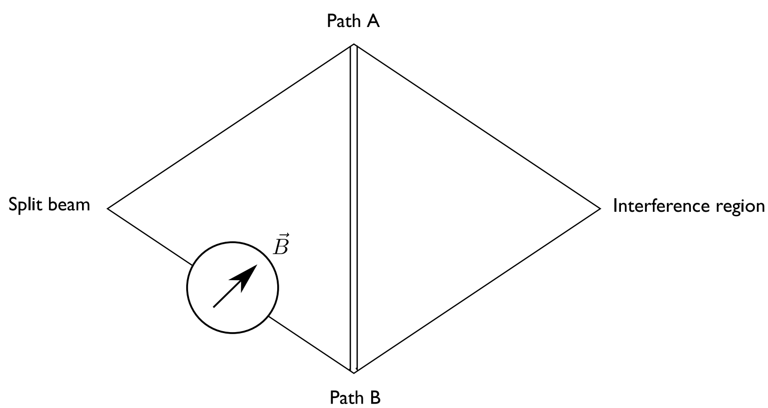
$$R(U(a,b)) = R(U(-a,-b)),$$

which implies that

$$SO(3) = SU(2)/\mathbb{Z}_2.$$

6.4 Realization of 4π rotation of spin$-1/2$

We have discussed the difference between $SU(2)$ and $SO(3)$ and found that the 2π rotation in $SU(2)$ and $SO(3)$ gives rise to a different final state. At this point, you might be wondering whether or not we can physically observe this difference between 2π and 4π rotations in a spin$-1/2$ system. We learn that 2π rotation of a spin$-1/2$ particle will introduce a phase factor of -1 to a state. The question is whether or not we can observe this phase change experimentally. Two experiments independently conducted in 1975 by Werner et al. [3] and Rauch et al. [4] were able to show that 4π (not 2π) rotations of the spin-1/2 particle using neutron interferometry preserve the original state.



A nearly monoenergetic thermal neutron beam is split into two paths A and B as shown in the figure. In Path B, the beam is directed through a region with non-zero magnetic field, where the Hamiltonian can be described by

$$\mathcal{H} = -\vec{\mu}_N \cdot \vec{B} = -\frac{g_N e\hbar}{2m_p c}\vec{\sigma} \cdot \vec{B}.$$

Hence, the beam in Path B will pick up an extra phase proportional to $e^{iHt/\hbar} = e^{\mp i\omega t/2}$, where it is assumed that $\vec{B}$ is along the $z-$axis and $\omega = \dfrac{g_N eB}{m_p c}$. t is a time spent in a region with non-zero $\vec{B}-$field.

At the interference region, where two beams combine again, the amplitude of the neutron beam arriving via Path B is equal to

$$c_2(t) = c_2(B=0)e^{\mp i\omega t/2} = e^{i\delta_2}e^{\mp i\omega t/2},$$

and on the other hand for Path A, where there is no change in the amplitude, the amplitude of the neutron is

$$c_1 = c_1(B=0) = e^{i\delta_1},$$

where δ_1 and δ_2 are phase factors for the beams in Paths A and B, respectively. The state at the interference region is, therefore, a superposition of the two beams.

$$\begin{aligned}
|\psi\rangle &= c_1\,|\phi\rangle + c_2 e^{\mp i\omega t/2}\,|\phi\rangle = \left(c_1 + c_2 e^{\mp i\omega t/2}\right)|\phi\rangle \\
\Rightarrow \quad \langle\psi|\psi\rangle &= \left(c_1 + c_2 e^{\mp i\omega t/2}\right)\left(c_1^* + c_2^* e^{\pm i\omega t/2}\right)\langle\phi|\phi\rangle \\
&= \left(|c_1|^2 + |c_2|^2 + c_1 c_2^* e^{\pm i\omega t/2} + c_1^* c_2 e^{\mp i\omega t/2}\right) \\
&= \left(2 + e^{\mp i\omega t/2 + i\delta} + e^{\pm i\omega t/2 - i\delta}\right) \\
&= 2\left(1 + \cos\left(\frac{\mp\omega t}{2} + \delta\right)\right) \\
\Rightarrow \quad \langle\psi|\psi\rangle &= 4\cos^2\left(\frac{\mp\omega t}{4} + \frac{\delta}{2}\right)
\end{aligned}$$

The intensity at the interference region shows a sinusoidal variation. t depends on a size of the region with non-zero magnetic field and is fixed, but we can vary B to observe the oscillation. Assume that the spin state of the neutron beam via Path B rotates back to the origin spin state after 4π rotation. The time T used to take the state back to the original state is given by

$$T = \frac{4\pi}{\omega} = \frac{4\pi m_p c}{e g_N \Delta B},$$

but T is also equal to the path length l inside the region with non-zero $\vec{B}-$field divided by the neutron velocity v

$$T = \frac{l}{v} = \frac{l}{\hbar k/m_p}.$$

Therefore, the value of ΔB that will take the spin state back to the original state is equal to

$$\Delta B = \frac{4\pi m_p c}{e g_N}\cdot\frac{\hbar k}{l m_p} = \frac{4\pi\hbar c k}{e g_n l}.$$

Therefore, a spin$-1/2$ particle requires 4π rotation to get back to the original state. These two experiments in 1975 showed for the time that the rotation group $SO(3)$ is insufficient to describe the rotation of a quantum mechanical state, and that in quantum mechanics the larger rotation group $SU(2)$ is required.

6.5 Representations of $SU(2)$

In the previous lecture, starting from the non-abelian group of rotation we have shown that the generators of rotation, which are the angular momentum operators, in quantum mechanics must satisfy the following commutator relation

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k.$$

Our goal in this lecture is to find the representations of these operators. We will start by defining

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

and

$$J_\pm = J_x \pm iJ_y.$$

Then, we can show that

$$\begin{aligned}
\left[J^2, J_i\right] &= 0 \quad i = x, y, z \\
[J_z, J_\pm] &= \pm\hbar J_\pm \\
[J_+, J_-] &= 2\hbar J_z,
\end{aligned}$$

which we can use to rewrite J^2 as

$$J^2 = J_z^2 + \frac{1}{2}\left(J_+J_- + J_-J_+\right) = J_z^2 + J_-J_+ + \hbar J_z.$$

To prove that $\left[J^2, J_i\right] = 0$ for $i = x, y$, and z, we will only choose the case where $i = z$.

$$\begin{aligned}
\left[J^2, J_z\right] &= \left[J_x^2 + J_y^2 + J_z^2, J_z\right] \\
&= J_x\left[J_x, J_z\right] + \left[J_x, J_z\right]J_x + J_y\left[J_y, J_z\right] + \left[J_,J_z\right]J_y \\
&= J_x\left(-i\hbar J_y\right) + \left(-i\hbar J_y\right)J_x + J_y\left(i\hbar J_x\right) + i\hbar J_x J_y \\
\Rightarrow \quad \left[J^2, J_z\right] &= 0.
\end{aligned}$$

Similar results can be obtained for $\left[J^2, J_x\right]$ and $\left[J^2, J_y\right]$. For the rest of the commutator relation, we can show that

$$\begin{aligned}
\left[J_z, J_\pm\right] &= \left[J_z, J_x \pm iJ_y\right] \\
&= \left[J_z, J_x\right] \pm i\left[J_z, J_y\right] \\
&= i\hbar J_y \pm i\left(-i\hbar J_x\right) \\
&= \hbar\left(iJ_y \pm J_x\right) \\
&= \pm\hbar\left(J_x \pm iJ_y\right) \\
\Rightarrow \quad \left[J_z, J_\pm\right] &= \pm\hbar J_\pm,
\end{aligned}$$

and

$$\begin{aligned}
\left[J_+, J_-\right] &= \left[J_x + iJ_y, J_x - iJ_y\right] \\
&= -i\left[J_x, J_y\right] + i\left[J_y, J_x\right] \\
&= -i\left(\hbar J_z\right) + i\left(-i\hbar J_z\right) \\
\Rightarrow \quad \left[J_+, J_-\right] &= 2\hbar J_z.
\end{aligned}$$

Now, since $\left[J^2, J_z\right] = 0$ (J^2 and J_z are compatible), we can find the simultaneous eigenstates of J^2 and J_z, where we will denote eigenvalues of J^2 and J_z by a and b, respectively. Hence, we have the following eigenvalue problems

$$J^2\left|a, b\right\rangle = a\left|a, b\right\rangle \quad \text{and} \quad J_z\left|a, b\right\rangle = b\left|a, b\right\rangle.$$

Next we have to solve these eigenvalue problems for the allowed values of a and b. First let us consider

$$\langle a, b|J^2|a, b\rangle = \left\langle a, b\,\middle|\, J_z^2 + \frac{1}{2}\left(J_+J_- + J_-J_+\right) \,\middle|\, a, b\right\rangle,$$

but $\langle a, b|J_+J_-|a, b\rangle$ and $\langle a, b|J_-J_+|a, b\rangle$ must be greater than zero since J_+ is an adjoint of J_- and vice versa. That is, $\langle a, b|\, J_+$ is a dual state of $J_-\left|a, b\right\rangle$ and $\langle a, b|\, J_-$ is a dual state of $J_+\left|a, b\right\rangle$.

$$\Rightarrow \langle a, b|J^2|a, b\rangle \geqslant \langle a, b|J_z^2|a, b\rangle,$$

which implies that

$$a \geqslant b^2$$

The commutator relations $\left[J_z, J_\pm\right] = \pm\hbar J_\pm$ suggested that $J_\pm$ is a "ladder operator" that raises (J_+) or lowers (J_-) the eigenvalue of J_z by a unit of $\hbar$. In order to show this, we consider

$$\begin{aligned}
J_z\left(J_\pm\left|a, b\right\rangle\right) &= \left(\left[J_z, J_\pm\right] + J_\pm J_z\right)\left|a, b\right\rangle \\
&= \left(\pm\hbar J_\pm + J_\pm b\right)\left|a, b\right\rangle \\
&= \left(b \pm \hbar\right)\left(J_\pm\left|a, b\right\rangle\right) \\
\Rightarrow \quad J_\pm\left|a, b\right\rangle &= C_\pm(a, b)\left|a, b \pm \hbar\right\rangle.
\end{aligned}$$

Since $a \geqslant b^2$, there must be a maximum $b_{\max} = j\hbar$ (where we will show later that j is half-integer), which implies that there must be a state with $b_{\max}$ such that

$$J_+ \left|a, b_{\max}\right\rangle = 0,$$

since there is no state with b greater than $b_{\max}$. Therefore, we have

$$\langle a, b_{\max}|J_- J_+|a, b_{\max}\rangle = \left|C_+(a, b_{\max})\right|^2 = 0,$$

but

$$\begin{aligned}
\langle a, b_{\max}|J_- J_+|a, b_{\max}\rangle &= \left\langle a, b_{\max} \,\middle|\, J_x^2 + J_y^2 + iJ_xJ_y - iJ_yJ_x \,\middle|\, a, b_{\max}\right\rangle \\
&= \left\langle a, b_{\max} \,\middle|\, J^2 - J_z^2 - \hbar J_z \,\middle|\, a, b_{\max}\right\rangle \\
\Rightarrow \quad 0 = \left|C_+(a, a, b_{\max})\right|^2 &= a - b_{\max}^2 - \hbar b_{\max} \\
\Rightarrow \quad a &= b_{\max}^2 + \hbar b_{\max} = \hbar^2 j^2 + \hbar^2 j \\
\Rightarrow \quad a &= \hbar^2 j(j+1)
\end{aligned}$$

Similarly, from $a \geqslant b^2$, there must be a minimum $b_{\min}$ such that

$$J_- \left|a, b_{\min}\right\rangle = 0.$$

Therefore, we have

$$\begin{aligned}
\langle a, b_{\min}|J_+ J_-|a, b_{\min}\rangle &= \left\langle a, b_{\min} \,\middle|\, J_x^2 + J_y^2 - iJ_xJ_y + iJ_yJ_x \,\middle|\, a, b_{\min}\right\rangle \\
&= \left\langle a, b_{\min} \,\middle|\, J^2 - J_z^2 + \hbar J_z \,\middle|\, a, b_{\min}\right\rangle \\
\Rightarrow \quad 0 = \left|C_-(a, a, b_{\min})\right|^2 &= a - b_{\min}^2 + \hbar b_{\min}
\end{aligned}$$

Hence, we have the relationship between $b_{\min}^2$ and $b_{\max}^2$ from

$$\begin{aligned}
a - b_{\min}^2 + \hbar b_{\min} &= a - b_{\max}^2 - \hbar b_{\max} \\
\Rightarrow \quad b_{\min} &= -b_{\max}. \\
\Rightarrow \quad 2b_{\max} &= n\hbar,
\end{aligned}$$

where n is an integer. Therefore, $j = \dfrac{n}{2}$ can be integer (if n is even) or half-odd-integer (if n is odd).

So, now we have solve the eigenvalue problems for a and b. Given j we obtain

$$\boxed{a = \hbar^2 j(j+1)}$$

and

$$\boxed{b = m\hbar},$$

where $m = -j, -j+1, \cdots, j-1, j$. Therefore, instead of labeling the eigenstates using a and b, we will switch to j and m

$$|a, b\rangle \rightarrow |j, m\rangle$$

with

$$\boxed{J^2 \left|j, m\right\rangle = \hbar^2 j(j+1) \left|j, m\right\rangle \quad \text{and} \quad J_z \left|j, m\right\rangle = m\hbar \left|j, m\right\rangle}$$

We call j the *total angular momentum quantum number* and m the *magnetic quantum number*.

Next, we want to find the representations for J_- and J_+ by considering $C_\pm(j,m)$. From

$$\begin{aligned} |C_\pm(a,b)|^2 &= a - b^2 \mp \hbar b \\ \Rightarrow \quad |C_\pm(j,m)|^2 &= \hbar^2 \left(j(j+1) - m^2 \mp m\right) = \hbar^2 (j \mp m)(j \pm m + 1) \\ \Rightarrow \quad C_\pm(j,m) &= \hbar\sqrt{(j \mp m)(j \pm m + 1)}. \end{aligned}$$

Therefore,

$$\boxed{J_\pm |j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle}$$

Therefore, the matrix elements for J^2, J_z, and $J_\pm$ in the $|j,m\rangle$ basis are given by

$$\begin{aligned} \langle j', m'|J^2|j,m\rangle &= \hbar^2 j(j+1)\delta_{j',j}\delta_{m',m} \\ \langle j', m'|J_m|j,m\rangle &= m\hbar\delta_{j',j}\delta_{m',m} \\ \langle j', m'|J_\pm|j,m\rangle &= \hbar\sqrt{(j \mp m)(j \pm m + 1)}\delta_{j',j}\delta_{m',m\pm 1} \end{aligned}$$

6.6 Orbital angular momentum

We will use the same trick that we have used before to find the eigen wavefunctions for the harmonic oscillators. First, we find the state with the largest value of m_l, that is, $m_l = l$. Since $m_l = l$ is the largest possible value of m_l given l, the state $|l, m_l = l\rangle = |l,l\rangle$ cannot be "raised" by L_+ anymore, that is,

$$L_+ |l,l\rangle = 0.$$

Next we will have to write the state $L_+ |l,l\rangle$ in the coordinate basis, which we will choose to write it in the spherical coordinate. We hence have to find an expression for L_+ in the spherical coordinate.

$$L_+ = L_x + iL_y$$

but we know that in the coordinate basis

$$\begin{aligned} L_x &\rightarrow y\left(-i\hbar\frac{\partial}{\partial z}\right) - z\left(-i\hbar\frac{\partial}{\partial y}\right) \\ L_y &\rightarrow z\left(-i\hbar\frac{\partial}{\partial x}\right) - x\left(-i\hbar\frac{\partial}{\partial z}\right) \end{aligned}$$

Hence,

$$\begin{aligned} L_\pm &= L_x \pm iL_y \\ &\rightarrow (y \mp ix)\left(-i\hbar\frac{\partial}{\partial z}\right) - z\left[\hbar\left(-i\frac{\partial}{\partial y} \mp \frac{\partial}{\partial x}\right)\right] \end{aligned}$$

Now you know the trick. Since it is easier to work in the spherical coordinate, we will change from the cartesian coordinate to the spherical coordinate. However, since the derivation is quite lengthy and does not add any new physics idea so I will not reproduce it here. Instead, I encourage you to read the beginning of Chapter 11 in Gasiorowicz [5]. The result of the coordinate transformation is

$$L_\pm \rightarrow \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right).$$

Similarly, we can find the expressions for L_x and L_y in the coordinate basis:

$$
\begin{aligned}
L_x &\rightarrow i\hbar\left(\sin\phi\frac{\partial}{\partial\theta}+\cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)\\
L_y &\rightarrow i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta}+\sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)
\end{aligned}
$$

Therefore, Eq. 6.1 written in the coordinate basis becomes

$$
\begin{aligned}
\langle\theta,\phi|L_+|l,l\rangle &= 0\\
\Rightarrow\quad \left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)Y_l^l(\theta,\phi) &= 0,
\end{aligned}
$$

where $\langle\theta,\phi|l,m_l\rangle = Y_l^{m_l}(\theta,\phi)$ is the eigen wavefunction in the coordinate basis of L^2 and L_z. Note that we will ignore the r dependence of the wave function for now and only consider the θ and ϕ dependence.

In order to solve this differential equation, we first have to recognize that from the two dimensional rotation we know that the eigen wavefunction of L_z, which depends only on ϕ is equal to $e^{im_l\phi}$, which has to remain the same in the three dimensional case as well. Therefore, we will write $Y_l^{m_l}(\theta,\phi)$ as

$$
Y_l^{m_l}(\theta,\phi) = \Theta_l^{m_l}(\theta)e^{im_l\phi}.
$$

The above differential equation hence becomes

$$
\begin{aligned}
\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)\Theta_l^l(\theta)e^{il\phi} &= 0\\
\Rightarrow\quad \left(\frac{\partial}{\partial\theta}-l\cot\theta\right)\Theta_l^l(\theta) &= 0\\
\Rightarrow\quad \frac{d\Theta_l^l}{\Theta_l^l} = l\frac{\cos\theta\ d\theta}{\sin\theta} &= l\frac{d(\sin\theta)}{\sin\theta}\\
\Rightarrow\quad \Theta_l^l(\theta) &= C\left(\sin\theta\right)^l
\end{aligned}
$$

Therefore, $Y_l^l(\theta,\phi)$ becomes

$$
Y_l^l(\theta,\phi) = N_{ll}\left(\sin\theta\right)^l e^{il\phi}.
$$

where N is the normalization factor such that $\int_{-1}^{1}\int_0^{2\pi} d(\cos\theta)d\phi\ \left|Y_l^l(\theta,\phi)\right| = 1$, which gives

$$
N_{ll} = \left[\frac{(2l+1)\,(2l)!}{4\pi}\right]^{1/2}\frac{1}{2^l l!},
$$

At the end, we can write the state with the largest value of m_l for a given value of l as

$$
Y_l^l(\theta,\phi) = (-1)^l\left[\frac{(2l+1)\,(2l)!}{4\pi}\right]^{1/2}\frac{1}{2^l l!}\left(\sin\theta\right)^l e^{il\phi}.
$$

where the factor $(-1)^{m_l}$ (in this case $(-1)^l$ since $m_l = l$) called the *Condon-Shortley phase* is a consequence of the operators L_- and L_+, which move the states labelled by m_l down and up, respectively, as we will see later.

Now in order to obtain other eigen wave functions with different value of m_l, all we have to do is to apply L_-, that is,

$$
L_-\left|l,l\right\rangle = \hbar\left[(l+1)\,(1)\right]^{1/2}\left|l,l-1\right\rangle = \hbar(2l)^{1/2}\left|l,l-1\right\rangle
$$

In the coordinate basis, we have

$$\begin{aligned}
\langle\theta,\phi|L_-|l,l\rangle &= \hbar(2l)^{1/2}\langle\theta,\phi|l,l-1\rangle \\
\Rightarrow\quad Y_l^{l-1}(\theta,\phi) &= \frac{1}{\hbar(2l)^{1/2}}\left[(-1)\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\phi}\right)\right]Y_l^l. \\
&= (-1)^{l-1}N_{l,l-1}\left[e^{-i\phi}\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\phi}\right)\right](\sin\theta)^l e^{il\phi} \\
&= (-1)^{l-1}N_{l,l-1}\left[e^{i(l-1)\phi}\left(\frac{\partial}{\partial\theta}+l\cot\theta\right)\right](\sin\theta)^l \\
&= (-1)^{l-1}N_{l,l-1}e^{i(l-1)\phi}\frac{1}{(\sin\theta)^l}\frac{d}{d\theta}(\sin\theta)^{2l}
\end{aligned}$$

In the last step we use the following identity:

$$\left(\frac{\partial}{\partial\theta}+l\cot\theta\right)f(\theta)=\frac{1}{(\sin\theta)^l}\frac{d}{d\theta}(\sin\theta)^l f(\theta).$$

For our case, $f(\theta)=(\sin\theta)^l$Note that one action of L_- will introduce one factor of -1; this is the reason why we need to multiply by the phase factor of $(-1)^{m_l}$. We can continue this process until we obtain all $2l+1$ states of the eigen wavefunctions for a given value of l. For example, for $Y_l^{l-2}(\theta,\phi)$, we have

$$Y_l^{l-2}(\theta,\phi)=(-1)^{l-2}N_{l,l-2}\frac{e^{i(l-2)\phi}}{(\sin\theta)^{l-1}}\frac{d}{d\theta}\left[\frac{1}{(\sin\theta)}\frac{d}{d\theta}(\sin\theta)^{2l}\right]$$

Or, we can rewrite both $Y_l^{l-1}(\theta,\phi)$ and $Y_l^{l-2}(\theta,\phi)$ in terms of $\cos\theta$ and $d(\cos\theta)$, noting that

$$\sin^2\theta=1-\cos^2\theta$$

and

$$\frac{d}{d(\cos\theta)}=\frac{-1}{\sin\theta}\frac{d}{d\theta}.$$

Hence, for $m\geqslant 0$, we have

$$\begin{aligned}
Y_l^{l-1} &= (-1)^{l-1}N_{l,l-1}e^{i(l-1)\phi}\frac{1}{(\sin\theta)^{l-1}}\frac{d}{d(\cos\theta)}(\sin\theta)^{2l} \\
Y_l^{l-2} &= (-1)^{l-2}N_{l,l-2}e^{i(l-2)\phi}\frac{1}{(\sin\theta)^{l-2}}\frac{d^2}{d(\cos\theta)^2}(\sin\theta)^{2l} \\
&\vdots \\
Y_l^{m_l} &= (-1)^{m_l}N_{l,m_l}e^{im_l\phi}\frac{1}{(\sin\theta)^{m_l}}\frac{d^{l-m_l}}{d(\cos\theta)^{l-m_l}}(\sin\theta)^{2l}
\end{aligned}$$

Again the normalization factor, N_{l,m_l} can be calculated by requiring that

$$\int_{-1}^{1}\int_0^{2\pi} d(\cos\theta)d\phi\;|Y_l^{m_l}(\theta,\phi)|=1.$$

However, the integration is quite complicated so I recommend that if you want to try it for fun, you might want to try it with Mathematica. Finally, the normalized eigen wave functions of the state $|l,m\rangle$ written in the coordinate basis for $m_l\geqslant 0$ can be expressed in the following form

$$Y_l^{m_l}(\theta,\phi)=(-1)^l\left[\frac{(2l+1)(2l)!}{4\pi}\right]^{1/2}\frac{1}{2^l l!}\left[\frac{(l+m_l)!}{(2l)!(l-m_l)!}\right]^{1/2}e^{im_l\phi}\frac{1}{(\sin\theta)^{m_l}}\frac{d^{l-m_l}}{d(\cos\theta)^{l-m_l}}(\sin\theta)^{2l}$$

or alternatively we can write it as

$$Y_l^{m_l}(\theta,\phi) = (-1)^{m_l}\left[\frac{(2l+1)}{4\pi}\frac{(l-m_l)!}{(l+m_l)!}\right] e^{im_l\phi} P_l^{m_l}(\cos\theta),$$

where $P_l^{m_l}(x)$, $(0 \leqslant m_l \leqslant l)$, is called the *associated Legendre polynomials.* For $m_l \leqslant 0$, we have

$$Y_l^{-m_l} = (-1)^{m_l}\left(Y_l^{m_l}\right)^*$$

We call the functions $Y_l^{m_l}$ the *spherical harmonics*. The first few terms of the spherical harmonics are

$$\begin{aligned}
Y_0^0 &= \frac{1}{\sqrt{4\pi}} \\
Y_1^{\pm 1} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \\
Y_1^0 &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_2^{\pm 2} &= \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi} \\
Y_2^{\pm 1} &= \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi} \\
Y_2^0 &= \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)
\end{aligned}$$

Note that by just looking at these eigen wave functions, we can readily read the value of l and m_l from the combined power of $\cos\theta$ and $\sin\theta$ and the pre-factor in front of $i\phi$ in the exponential term.

In order to understand the spherical harmonics functions better, it is a good idea to express them in the (spherical) coordinate basis. The figure below shows the spherical harmonics for $l = 2$. As you can see, if m_l is large (in this case the largest value of m_l is equal to 2), the wave function is mostly confined in the $x - y$ plane, which is what we would expect if the angular momentum along the z-axis is large; remember that $m_l\hbar$ denotes the z-component of the angular momentum. In the other extreme, if $m_l = 0$, that is, the z-component of the angular momentum is small, the wave function will be along the z-axis. This behavior is consistent with a classical picture, where $\vec{L} = \vec{r}\times\vec{p}$, that is, if the wave function is localized along the z-aixs, $\vec{r}$ is small and hence the angular momentum along z is also small. On the other hand, if the wave function spreads out in the $x - y$ plane, then $\vec{r}$ becomes large and hence the angular momentum along z is large.

We can also solve for the spherical harmonics in terms of L^2 and L_z in the coordinate basis directly as well. In fact, if you have solved the hydrogen atom problem in other class, you have most likely done those calculations already. In the coordinate basis, L^2 and L_z become

$$\begin{aligned}
L^2 &\rightarrow (-\hbar^2)\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \\
L_z &\rightarrow -i\hbar\frac{\partial}{\partial\phi}
\end{aligned}$$

That is, we will obtain the following differential equations

$$\begin{aligned}
\langle\theta,\phi|L^2|l,m_l\rangle &= l(l+1)\hbar^2\,\langle\theta,\phi|l,m_l\rangle \\
\Rightarrow\quad (-\hbar^2)\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)Y_l^{m_l}(\theta,\phi) &= l(l+1)\hbar^2 Y_l^{m_l}
\end{aligned}$$

and

$$
\begin{aligned}
\langle\theta,\phi|L_|l,m_l\rangle &= m_l\hbar\,\langle\theta,\phi|l,m_l\rangle \\
\Rightarrow\qquad -i\hbar\frac{\partial}{\partial\phi}Y_l^{m_l}(\theta,\phi) &= m_l\hbar^2 Y_l^{m_l}.
\end{aligned}
$$

To solve these equations we can use the separation of variables and write

$$
Y_l^{m_l}(\theta,\phi) = \Theta_l^{m_l}(\theta)\Phi_{m_l}(\phi).
$$

However, we will not solve for $Y_l^{m_l}$ but instead we will use these differential equations to simplify our future calculations in finding the eigenenergies and eigenwavefunctions of the hydrogen atom.

6.7 Rotation operators

As suggested by what we have discussed in the previous section, for each j we can construct an irreducible representation of the $SU(2)$ algebra, where the Hilbert space H_j is spanned by a set

$$
\{|j,m\rangle\,,\text{where } m=-j,-j+1,\cdots,j-1,j\}\,.
$$

Then we can use the representation of J to construct the representation of the rotation group $\mathcal{D}(R)$, where R is rotation in a classical sense. The matrix elements of $\mathcal{D}(R)$ in the basis $|j,m\rangle$ is given by

$$
\mathcal{D}^{(j)}_{m',m}(R) = \langle j,m'|e^{-i\left(J^{(j)}\cdot\hat{n}\right)\phi/\hbar}|j,m\rangle\,.
$$

This is called *Wigner functions.*

Since J^2 commutes with J_i for all i's, $e^{-i\left(J^{(j)}\cdot\hat{n}\right)\phi/\hbar}$ does not change the eigenvalue of J^2. Therefore, $\mathcal{D}^{(j)}_{m',m}(R)$ form a $(2j+1)\times(2j+1)$ matrix with a definite value of j, and this matrix cannot be broken into smaller blocks. We refer to this matrix as the $(2j+1)-$*dimensional irreducible representation of* $\mathcal{D}(R)$.

Our task is to figure out the representation of $\mathcal{D}(R)$. Let us look at some of the simple cases where j is small.

1. $j=0$ (trivial case)

   The only eigenstate is $|j,m\rangle = |0,0\rangle$, and

   $$
   J^2\,|0,0\rangle = J_z\,|0,0\rangle = J_\pm\,|0,0\rangle = 0.
   $$

   The rotation operator is

   $$
   \mathcal{D}^(0)(R)\,|0,0\rangle = |0,0\rangle\,,
   $$

   that is, for all R

   $$
   \mathcal{D}^{(0)}(R) = \mathbb{1}.
   $$

2. $j=1$ (spin$-1/2$ systems)

   The eigenstates are

   $$
   |j,m\rangle = |1/2,\pm1/2\rangle \equiv |S_z=\pm1\rangle \equiv |\pm\rangle\,.
   $$

   In this basis, the angular momentum operators are

   $$
   J_i = S_i = \frac{\hbar}{2}\sigma_i,
   $$

where $i = x, y, z$, and σ_i are the Pauli matrices. The relevant angular momentum operators are

$$J^2 = \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad J_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad J_+ = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad J_- = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

The rotation operator is given by

$$\mathcal{D}^{(1/2)}(\hat{n}, \phi) = e^{-i(\hat{n}\cdot\vec{\sigma}\phi/2)} = \mathbb{1}\cos\frac{\phi}{2} - i\left(\hat{n}\cdot\vec{\sigma}\right)\sin\frac{\phi}{2}.$$

As you can see, the rotation operator becomes a bit more complicated.

3. $j = 1$ (spin-1 systems)

   The eigenstates are

$$|j, m\rangle = |1, \pm 1\rangle \text{ and } |1, 0\rangle.$$

   The angular momentum operators are

$$J_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$$

   and

$$J_+ = \hbar\sqrt{2}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}, \quad J_- = \hbar\sqrt{2}\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

   For the rotation around an axis defined by a unit vector $\hat{n}$, the rotation operator is given by

$$\mathcal{D}^{(1)}(\hat{n}, \phi) = e^{-i\vec{J}\cdot\hat{n}\phi/\hbar},$$

   which can be rewritten as

$$\mathcal{D}^{(1)}(\hat{n}, \phi) = (1) - \left(\frac{J_n}{\hbar}\right)^2(1 - \cos\phi) - \frac{iJ_n}{\hbar}\sin\phi,$$

   where $J_n \equiv \vec{J}\cdot\hat{n}$. However, since the non-trivial rotation is the rotation about the $y-$axis, we will only consider this rotation. The rotation operation is given by

$$\mathcal{D}^{(1)}(\hat{y}, \phi) = e^{-iJ_y\phi/\hbar}.$$

   Similar to the $S = 1/2$ case, we can expand the exponential using the Taylor expansion, keeping in mind that

$$\left(\frac{J_y^{(1)}}{\hbar}\right)^3 = \frac{J_y^{(1)}}{\hbar}.$$

   We find that

$$\mathcal{D}^{(1)}(\hat{y}, \phi) = \mathbb{1} - \left(\frac{J_y}{\hbar}\right)^2(1 - \cos\phi) - \frac{iJ_y}{\hbar}\sin\phi.$$

   We can see that $\mathcal{D}^{(j)}(R)$ can become very complicated for large j. Therefore, we need a better method to write down $\mathcal{D}^{(j)}(R)$.

6.8 Schwinger's oscillator model of angular momentum

Consider two types of simple harmonic oscillators (SHO), which we will call + type and − type. We define operators for both types as follows:

$$\begin{array}{ll} +\ \text{type} & -\ \text{type} \\ N_+ = a_+^\dagger a_+ & N_+ = a_-^\dagger a_- \\ \left[a_+, a_+^\dagger\right] = 1 & \left[a_-, a_-^\dagger\right] = 1 \\ \left[N_+, a_+^\dagger\right] = a_+^\dagger & \left[N_-, a_-^\dagger\right] = a_-^\dagger \\ \left[N_+, a_+\right] = -a_+ & \left[N_-, a_-\right] = -a_- \end{array} \tag{6.1}$$

where $a_\pm^\dagger$ and $a_\pm$ are raising and lowering operators and $N_\pm$ are the number operators for $\pm$ type. For different types of SHO, the commutators are zero since we assume that there is no coupling between the two oscillators. That is,

$$\left[a_+, a_-^\dagger\right] = \left[a_-, a_+^\dagger\right] = \left[a_+, a_-\right] = \left[a_+^\dagger, a_-^\dagger\right] = 0,$$

and in particular

$$\left[N_+, N_-\right] = 0.$$

Therefore, since N_+ and N_- are compatible, we can label the eigenstates using the eigenvalues of N_+ and N_-, giving the simultaneous eigenstates $|n_+, n_-\rangle$, where

$$N_+ |n_+, n_-\rangle = n_+ |n_+, n_-\rangle, \qquad N_- |n_+, n_-\rangle = n_- |n_+, n_-\rangle,$$

and we know that

$$\begin{aligned} a_+^\dagger |n_+, n_-\rangle &= \sqrt{n_+ + 1}\, |n_+ + 1, n_-\rangle, \quad a_-^\dagger |n_+, n_-\rangle = \sqrt{n_- + 1}\, |n_+, n_- + 1\rangle \\ a_+ |n_+, n_-\rangle &= \sqrt{n_+}\, |n_+ - 1, n_-\rangle, \qquad\quad a_- |n_+, n_-\rangle = \sqrt{n_-}\, |n_+, n_- - 1\rangle \end{aligned}$$

We can construct $|n_+, n_-\rangle$ from $|0, 0\rangle$ using $a_+^\dagger$ and $a_-^\dagger$ as follows

$$|n_+, n_-\rangle = \frac{\left(a_+^\dagger\right)^{n_+}}{\sqrt{n_+!}} \frac{\left(a_-^\dagger\right)^{n_-}}{\sqrt{n_-!}} |0, 0\rangle$$

Now, we can make a connection between $a_\pm^\dagger$, $a_\pm$, and the angular momentum operators by defining the angular momentum operators in terms of $a_\pm^\dagger$ and $a_\pm$. Note that we only have to define $J_\pm$ and J_z, and then check that all commutator relations for J are satisfied.

$$\begin{aligned} J_+ &= \hbar a_+^\dagger a_- \\ J_- &= \hbar a_-^\dagger a_+ \\ J_z &= \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) = \frac{\hbar}{2}\left(N_+ - N_-\right) \end{aligned}$$

We need to show that

$$\begin{aligned} \left[J_z, J_\pm\right] &= \pm\hbar J_\pm \\ \left[J_+, J_-\right] &= 2\hbar J_z \\ \left[J^2, J_z\right] &= 0 \end{aligned}$$

For example, first consider

$$\begin{aligned}
[J_z, J_\pm] &= \frac{\hbar^2}{2}\left[a_+^\dagger a_+ - a_-^\dagger, a_-, a_+^\dagger a_-\right] \\
&= \frac{\hbar^2}{2}\left(\left[a_+^\dagger a_+, a_+^\dagger a_-\right] - \left[a_-^\dagger a_-, a_+^\dagger a_-\right]\right)
\end{aligned}$$

but

$$\begin{aligned}
\left[a_+^\dagger a_+, a_+^\dagger a_-\right] &= a_+^\dagger a_+ a_+^\dagger a_- - a_+^\dagger a_- a_+^\dagger a_+ \\
&= a^\dagger\left(a_+^\dagger + 1\right) a_- - a_+^\dagger a_+^\dagger a_+ a_- \\
&= a_+^\dagger a_+^\dagger a_+ a_+ + a_+^\dagger a_- - a_+^\dagger a_+^\dagger a_+ a_- \\
\left[a_+^\dagger a_+, a_+^\dagger a_-\right] &= = a_+^\dagger a_-
\end{aligned}$$

Similarly,

$$\left[a_-^\dagger a_-, a_+^\dagger a_-\right] = -a_+^\dagger a_-.$$

Therefore,

$$[J_z, J_\pm] = \frac{\hbar^2}{2}\left(a_+^\dagger a_- - \left(-a_+^\dagger a_-\right)\right) = \hbar^2 a_+^\dagger a_- = \hbar J_+.$$

I will leave as an exercise for you to prove the rest of the above commutator relations. Next let us consider the operator J^2. We can write J^2 as

$$J^2 = J_z^2 + \frac{1}{2}\left(J_+ J_- + J_- J_+\right),$$

where

$$\begin{aligned}
J_+ J_- &= \hbar^2 a_+^\dagger a_+ = \hbar^2 a_+^\dagger\left(a_-^\dagger a_- + 1\right) a_+ \\
&= \hbar^2\left(a_+^\dagger a_+ + a_+^\dagger a_-^\dagger a_- a_+\right) \\
&= \hbar^2\left(a_+^\dagger a_+ + \left(a_-^\dagger a_-\right)\left(a_+^\dagger a_+\right)\right) \\
\Rightarrow \quad J_+ J_- &= \hbar^2\left(N_+ + N_- N_+\right),
\end{aligned}$$

and similarly,

$$\Rightarrow \quad J_- J_+ = \hbar^2\left(N_- + N_+ N_-\right).$$

Therefore,

$$\begin{aligned}
\Rightarrow \quad J^2 &= \frac{\hbar^2}{4}\left(N_+ - N_-\right)^2 + \frac{\hbar^2}{2}\left(N_+ + N_- + 2N_+ N_-\right) \\
&= \frac{\hbar^2}{2}\left(\frac{N_+^2}{2} - N_+ N_- + \frac{N_-^2}{2} + \left(N_+ + N_-\right) + 2N_+ N_-\right) \\
&= \frac{\hbar^2}{2}\left(\frac{1}{2}\left(N_+^2 + 2N_+ N_- + N_-^2\right) + \left(N_+ + N_-\right)\right).
\end{aligned}$$

Let $N = N_+ + N_-$,

$$\Rightarrow \quad J^2 = \frac{\hbar^2}{2}\left(\frac{1}{2}N^2 + N\right) = \hbar^2 \frac{N}{2}\left(\frac{N}{2} + 1\right),$$

which implies that $j = \frac{n}{2}$, where $n = n_+ + n_-$ is an integer and eigenvalue of N, which is a total number of "spins" (including both "spin-up" created by $a_+^\dagger$ and "spin-up" created by $a_-^\dagger$).

The action of all momentum operators on $|n_+, n_-\rangle$ can be summarized as follows

$$\begin{aligned}
J^2 |n_+, n_-\rangle &= \hbar^2 \frac{n}{2}\left(\frac{n}{2}+1\right)|n_+, n_-\rangle \\
J_z |n_+, n_-\rangle &= \frac{\hbar}{2}(N_+ - N_-)|n_+, n_-\rangle = \frac{\hbar}{2}(n_+ + n_-)|n_+, n_-\rangle \\
J_+ |n_+, n_-\rangle &= \hbar a_+^\dagger a_- |n_+, n_-\rangle = \hbar\sqrt{n_-}\sqrt{n_+ + 1}\,|n_+ + 1, n_- - 1\rangle \\
J_- |n_+, n_-\rangle &= \hbar a_-^\dagger a_+ |n_+, n_-\rangle = \hbar\sqrt{n_+}\sqrt{n_- + 1}\,|n_+ - 1, n_- + 1\rangle
\end{aligned}$$

Hence, we can define j and m in terms of n_+ and n_- using

$$n_+ \to j + m \quad \text{and} \quad n_- \to j - m,$$

which gives

$$m = \frac{n_+ - n_-}{2}$$

while

$$j = \frac{n_+ + n_-}{2} \equiv \frac{n}{2},$$

which the same as what we have obtained before when considering J^2. Therefore, we can think of n_+ as a number of spin-ups and n_- as a number of spin-downs. $n = 2j$ is a total number of spin-1/2 particles while $2m = n_+ - n_-$ is a number of spin-ups minus a number of spin-downs. In this sense, $a_+^\dagger$ and $a_-^\dagger$ create one spin-up and one spin-down while a_+ and a_- annihilate one spin-up and one spin-down, respectively. We can then summarize the action of the momentum operators in this Schwinger's scheme as follows

- J^2 counts a total value of spins.
- J_z counts a value of spins along the $z-$axis.
- J_+ changes spin-down to spin-up (flip one spin-down). In other words, it annihilates one spin-down and creates one spin-up.
- J_- changes spin-up to spin-down (flip one spin-up). In other words, it annihilates one spin-up and creates one spin-down.

Now, we are ready to label the state using j and m instead of n_+ and n_-.

$$\begin{aligned}
|n_+, n_-\rangle &\longrightarrow |j, m\rangle \\
\Rightarrow \quad |j, m\rangle &= \frac{\left(a_+^\dagger\right)^{j+m}\left(a_-^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle,
\end{aligned}$$

where $|0\rangle$ is a vacuum state with $j = m = 0$. In particular,

$$|j, j\rangle = \frac{\left(a_+^\dagger\right)^{2j}}{\sqrt{(2j)!}}|0\rangle \quad \text{and} \quad |j, -j\rangle = \frac{\left(a_-^\dagger\right)^{2j}}{\sqrt{(2j)!}}|0\rangle.$$

Using this construction of $|j,m\rangle$ in terms of $a_\pm^\dagger$, we can construct the rotation operator $\mathcal{D}(R) = e^{-iJ_y\phi/\hbar}$ more easily. When $\mathcal{D}(R)$ acts on $|j,m\rangle$, we have

$$\begin{aligned}
\mathcal{D}(R)\,|j,m\rangle &= \mathcal{D}(R)\left[\left(\frac{\left(a_+^\dagger\right)^{j+m}\left(a_-^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\right)|0\rangle\right] \\
&= \frac{\left(\mathcal{D}(R)a_+^\dagger\mathcal{D}^{-1}(R)\right)^{j+m}\left(\mathcal{D}(R)a_-^\dagger\mathcal{D}^{-1}(R)\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\mathcal{D}(R)\,|0\rangle\,,
\end{aligned}$$

where $\mathcal{D}(R)\,|0\rangle = |0\rangle$. We obtain the above expression by realizing that the rotation of $|j,m\rangle$ is the same as the counter-rotation (transformation) of the operators $a_\pm^\dagger$ by the same rotation operator $\mathcal{D}(R)$. So, we have to consider the transformation of $a_\pm^\dagger$ by $\mathcal{D}(R)$.

$$\begin{aligned}
\mathcal{D}(R)a_\pm^\dagger\mathcal{D}^{-1}(R) &= e^{-iJ_y\phi/\hbar}a_\pm^\dagger e^{iJ_y\phi/\hbar} \\
&= a_\pm^\dagger + i\phi\left[\frac{-J_y}{\hbar}, a_\pm^\dagger\right] + \frac{i^2\phi^2}{2!}\left[\frac{-J_y}{\hbar},\left[\frac{-J_y}{\hbar}, a_\pm^\dagger\right]\right] + \cdots\,,
\end{aligned}$$

where we use the identity for $e^A B e^{-A}$, which is given by

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}A^n\{B\},$$

where

$$A^0\{B\} = B,\;\; A^1\{B\} = [A,B],\;\; A^2\{B\} = [A,[A,B]],\;\; A^3\{B\} = [A,[A,[A,B]]],\cdots$$

We note that $J_y = \frac{1}{2i}\left(J_+ - J_-\right) = \frac{\hbar}{2i}\left(a_+^\dagger a_- - a_-^\dagger a_+\right)$, and hence

$$\begin{aligned}
\left[\frac{-J_y}{\hbar}, a_+^\dagger\right] &= \frac{1}{2i}\left[a_-^\dagger a_+ - a_+^\dagger a_-, a_+^\dagger\right] = \frac{1}{2i}a_-^\dagger \\
\left[\frac{-J_y}{\hbar}, a_-^\dagger\right] &= \frac{1}{2i}\left[a_-^\dagger a_+ - a_+^\dagger a_-, a_-^\dagger\right] = \frac{-1}{2i}a_+^\dagger,
\end{aligned}$$

and

$$\begin{aligned}
\left[\frac{-J_y}{\hbar},\left[\frac{-J_y}{\hbar}, a_+^\dagger\right]\right] &= \frac{1}{2i}\left[\frac{-J_y}{\hbar}, a_-^\dagger\right] = \frac{1}{2i}\left(\frac{-1}{2i}\right)a_+^\dagger = \frac{1}{2^2}a_+^\dagger \\
\left[\frac{-J_y}{\hbar},\left[\frac{-J_y}{\hbar}, a_-^\dagger\right]\right] &= \frac{-1}{2i}\left[\frac{-J_y}{\hbar}, a_+^\dagger\right] = \frac{-1}{2i}\left(\frac{1}{2i}a_-^\dagger\right) = \frac{-1}{2^2}a_-^\dagger.
\end{aligned}$$

I hope that you can see the pattern and we can now calculate all higher terms. The result is

$$\begin{aligned}
\mathcal{D}(R)a_\pm^\dagger\mathcal{D}^{-1}(R) &= a_\pm^\dagger \pm \frac{\phi}{2}a_\mp^\dagger - \frac{1}{2!}\left(\frac{\phi}{2}\right)^2 a_\pm^\dagger \mp \frac{1}{3!}\left(\frac{\phi}{2}\right)^3 a_\mp^\dagger + \cdots \\
\Rightarrow\quad \mathcal{D}(R)a_\pm^\dagger\mathcal{D}^{-1}(R) &= a_\pm^\dagger\cos\frac{\phi}{2} \pm a_\mp^\dagger\sin\frac{\phi}{2}.
\end{aligned}$$

Therefore,

$$\mathcal{D}(R)\,|j,m\rangle = \frac{\left(a_+^\dagger\cos\frac{\phi}{2} + a_-^\dagger\sin\frac{\phi}{2}\right)^{j+m}\left(a_-^\dagger\cos\frac{\phi}{2} - a_+^\dagger\sin\frac{\phi}{2}\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\,|0\rangle\,.$$

We can rewrite this expression using the binomial expansion, where

$$(x+y)^N = \sum_{k=0}^{N} \frac{N!}{(N-k)!k!} x^{N-k} y^k,$$

by letting $x = a_+^\dagger \cos\frac{\phi}{2}$ and $y = a_-^\dagger \sin\frac{\phi}{2}$ for the first term and $x = -a_+^\dagger \sin\frac{\phi}{2}$ and $y = a_-^\dagger \cos\frac{\phi}{2}$ for the second term.

$$\begin{aligned}\mathcal{D}(R)\,|j,m\rangle &= \sum_{k=0}^{j+m}\sum_{l=0}^{j-m} \frac{(j+m)!(j-m)!}{(j+m-k)!k!(j-m-l)!l!} \frac{1}{\sqrt{(j+m)!(j-m)!}} \left(a_+^\dagger \cos\frac{\phi}{2}\right)^{j+m-k} \left(a_-^\dagger \sin\frac{\phi}{2}\right)^{k} \\ &\quad \times \left(-a_+^\dagger \sin\frac{\phi}{2}\right)^{j-m-l} \left(a_-^\dagger \cos\frac{\phi}{2}\right)^{l} |0\rangle\,.\end{aligned}$$

But we know that $\mathcal{D}(R)$ only mixes m and preserves j, and hence we can write $\mathcal{D}(R)$ as

$$\mathcal{D}(R)\,|j,m\rangle = \sum_{m'=-j}^{j} d^{(j)}_{m',m}\,|j,m\rangle = \sum_{m'=-j}^{j} d^{(j)}_{m',m} \frac{\left(a_+^\dagger\right)^{j+m}\left(a_-^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\,|0\rangle\,,$$

where $d^{(j)}_{m',m}$ is a $(2j+1)\times(2j+1)$ matrix that we want to find. To solve for $d^{(j)}_{m',m}$ all we have to do is to match coefficients of terms with the same order of $a_\pm^\dagger$ from both equations for $\mathcal{D}(R)\,|j,m\rangle$. By considering the exponent of the $a_+^\dagger$ term, we obtain

$$\begin{aligned} j+m-k+j-m+l &= j+m' \\ \Rightarrow \quad m' &= j-k-l, \end{aligned}$$

and similarly for $a_-^\dagger$, we have

$$\begin{aligned} k+l &= j-m' \\ \Rightarrow \quad m' &= j-k-l, \end{aligned}$$

So now we can choose to eliminate l or k. Let us take

$$l = j-k-m'$$

and eliminate l from the expression for $d^{(j)}_{m',m}$. Therefore, we obtain

$$\boxed{d^{(j)}_{m',m}(\hat{y},\phi) = \sum_{k=0}^{j+m} (-1)^{k-m+m'} \frac{\sqrt{(j+m)!(j-m)!}\sqrt{(j+m')!(j-m')!}}{(j+m-k)!k!(j-k-m')!(k-m+m')!} \left(\cos\frac{\phi}{2}\right)^{2j-2k+m-m'} \left(\sin\frac{\phi}{2}\right)^{2k-m-m'}}$$

This is a general solution for the matrix elements of the rotation operator $\mathcal{D}(R) = e^{-iJ_y\phi/\hbar}$ for any value of j, where $m, m' = -j, -j+1, \cdots, j-1, j$. Therefore, $d^{(j)}_{m',m}$ are the elements of the $(2j+1)\times(2j+1)$ matrix, a representation of the rotation operator $\mathcal{D}(R)$.

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