SCPY 371: Solid State Physics **1^{st} Semester 2021**

Lecture 10: Superconductivity

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For the first time, we will have to include the electron-electron interaction in order to be able to describe the phenomenon in this class. The first phenomenon that forces us to have to do that is superconductivity, which only arises when the interaction between electrons is considered. However, before we go into details how the electron-electron interaction can give rise to superconductivity. We will first learn about the physical properties that superconductors have. Note that in this class we will only consider the so called "conventional superconductor" and will ignore the other class of superconductors called "high-transition-temperature (High-T_c) superconductors", whose physics is currently unknown and intensively studied.

10.1 Physical Properties of Superconductors

The properties of the conventional superconductors that we have learned from doing countless experiments are the followings:

1. Superconductivity is found in non-magnetic metal. These metals enter the phase with zero resistivity at low temperatures, that is, $T < T_c$ where T_c is called a critical temperature. T_c can be as low as 0.01 K in some conventional superconductors and as high as 150 K in some high-T_c superconductors.
2. If there is a magnetic impurity in the superconductors (hereafter we will use superconductors to mean conventional superconductors for brevity), such as a small amount of iron (Fe), superconductivity will be suppressed or disappear.
3. The application of external magnetic field will suppressed or kill superconductivity. Furthermore, if the magnetic field exceeds a critical field H_c, superconductivity will disappear all together.

As we can see from above, superconductivity and magnetism are competing phases, that is, both cannot co-exist. This fact is also shown in the **Meissner effect**.

Meissner Effect Besides having zero resistivity at low temperatures, superconductors have another unique and important properties, that is, the total magnetic field inside it has to be zero ($B = 0$). In other words, if $H < H_c$ and $T < T_c$, magnetic field lines of the external magnetic field inside the superconductors will be pushed away. Suppose that a piece of the superconducting sample is thin and long. We can write the magnetic field inside as a sum of two terms; one from the external magnetic field and the other from the induced magnetic field from the superconductors themselves:

$$B = B_a + \mu_0 M = 0,$$

where B_a is the applied magnetic field and M is the induced magnetization. From this expression, we obtain

$$M = -\frac{B_a}{\mu_0}.$$

Therefore, the superconductors show the perfect diamagneitism (note that the direction of the magnetization is opposite to the direction of the applied field) and propel all external magnetic field line.

As we have mentioned, the Meissner effect is unique to the superconductors and will not occur in the perfect conductors. We will define the perfect conductors as a group of material with zero resistivity. Therefore, one can say that the Meissner effect is **NOT** a result of the zero resistivity. To illustrate this point, let us consider the transport and Maxwell's equations. First, we know that

$$\vec{E} = \rho \vec{j} = 0,$$

since $\rho = 0$ for the perfect conductors. However, from one of the Maxwell's equations, we know that

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} = 0,$$

that is, there is no change of magnetic field when the material change from the normal state to the perfect conducting state. However, this result contradicts the experimental found since we know that from the experiment the magnetic field inside the superconductors changes from B_a to zero when the temperature decreases below the critical temperature T_c. Therefore, being superconducting is different from being perfect conducting whereas for the former we have to include the Meissner effect as well. Furthermore, for the perfect conductors, the mean free path has to be infinite, which means the perfect conductor cannot create the eddy current to counter the external magnetic field forever. It turn out that the field can penetrate the sample at the rate of about 1 cm/hour.

Based on their response to the external magnetic field, we can catagorize the superconductors into two groups:

1. **Type-I superconductors**: In this case, the Meissner effect is always complete in the superconducting state. That is, if $H < H_c$, no magnetic field line can penetrate the sample and the sample is superconducting. On the other hand, if $H > H_c$, magnetic field lines can penetrate **all** sample and superconductivity disappears. In this case, H_c is usually very small in an order of $100 - 1,000$ Oe. This group contains pure specimens such as Pb, Hg, and etc.

2. **Type-II superconductors**: In this case, there are two important values of magnetic field. The first is called H_{c1}, which indicate that the Meissner effect is complete for $H < H_{c1}$. The other is called H_{c2}, which is larger than H_{c1}. If $H_{c1} < H < H_{c2}$, some magnetic field lines can penetrate the sample but the field does not kill superconductivity, that is, the sample remains in the superconducting state with only some part which the field can penetrate become a normal state. If $H > H_{c2}$, superconductivity disappears. In general, the materials in this group are alloys or transition metals that have high resistivity in the normal state, that is, the mean free path is short at high temperatures. Normally, H_{c2} can be very high, such as 41 T in alloys of Nb, Al and Ge or 54 T in $PbMo_6S_8$. Therefore, H_{c2} can be much larger than H_c. Type-II superconductors are used to build a high-field magnet since the heat loss from electric resistance is kept at minimal and they remains superconducting even at very high magnetic field.

Heat Capacity From the statistical mechanics class, we all know that heat capacity is related to entropy through the following integral expression:

$$S = \int_0^T \frac{C(T')}{T'} dT',$$

where we can measure the heat capacity as a function of temperature, and hence calculate the entropy. From the experiments, we found that the superconducting state has lower entropy than the normal state, that is, going through T_c the entropy of the system decreases rapidly. This results implies that in the superconducting state electrons are more ordered than those in the normal state. However, the difference in entropy between the normal state and the superconducting state is very small compared with the total entropy of the system, indicating that only a small fraction of electrons is superconducting while the rest are normal electrons. The experiments show that this entropy difference is only 10^{-4} k_B/atom.

We know from the previous chapter that in the normal state the heat capacity can be expressed by the following equation

$$\frac{C}{T} = A + BT^2,$$

where the first term on the right hand side is the electronic contribution and the second term the phonon contribution. In the superconducting state, the heat capacity will deviate from this formula. Since we are only interested in the heat capacity due to the superconducting electrons, we will only consider the difference of the heat capacity between the normal state and superconducting state; $C_{es} = C_{tot} - C_{phonon}$. From the experiments, we found that C_{es} is equal to

$$\frac{C_{es}}{T} = C_0 e^{-\Delta_s T_c/T}.$$

That is, the superconducting part of the heat capacity can be expressed in terms of the exponential function, which indicates that there is an energy gap in the system.

Energy Gap The heat capacity measurements give us a hint that there exists the energy gap in the superconducting state. This gap is, however, different from the gap in the insulators or semiconducting. Whereas the energy gap in the latter is due to the electron-lattice (or electron-nucleus interaction), the energy gap in the former is due to the electron-electron interaction. It turns out that in superconductors electrons can interaction with other electrons through phonons and the type of the interaction is attractive, which pairs two electrons in the momentum space with the wave vectors $\vec{k}$ and $-\vec{k}$. We will study this interaction in more detail when we get to the microscopic theory later in this class. Furthermore, we also found that the superconducting gap gives rise to an exponential form $e^{-E_g/2k_BT}$ instead of e^{-E_g/k_BT}, which indicates that two electrons are involved in forming the superconducting state.

Isotope effect The final hint to the mechanism behind superconductivity is the isotope effect. It was found that for Hg T_c changes from 4.185 K to 4.146 K when its atomic mass changes from 199.5 to 203.4. The relation between the atomic mass M and T_c can be expressed by

$$M^\alpha T_c = C,$$

where C and α are constant and can be different for different elements. This clue is very important since it leads to the obvious candidate (now that we know). That is, phonons must play an important role in superconductivity, since by changing the mass of the same element, we change the phonon energy as we have learned in this class. And that fact for some reason, affects T_c. From the equation above, we can see that increasing M will decrease T_c. In other words, decreasing the energy of phonons, which is a result of increasing M, will decrease T_c or will suppress superconductivity. Therefore, superconductivity must be related to lattice vibrations or phonons.

10.2 Phenomenological Theory

In this section, we will try to describe the results in the previous section using the phenomenological theory, which means that we will not attempt to explain the underlying mechanism or microscopic theory of the phenomena.

London equation We will start by considering the phenomenon of the penetration of the magnetic flux. The Meissner effect indicates that $\chi = -1$, perfect diamagnetism for superconductors. We learned in the previous section that we cannot use the equation $\vec{j} = \sigma\vec{E}$. Instead, we will suppose that the current density $\vec{j}$ is proportional to the vector potential $\vec{A}$, that is,

$$\vec{j} = -\frac{1}{\mu_0 \lambda_L^2}\vec{A},$$

where λ_L is a constant called London's penetration dept. The form of the constant term will be clear in a minute. Next, let us take $\nabla\times$ on both sides of the equation

$$\nabla\times\vec{j} = -\frac{1}{\mu_0\lambda_L^2}\nabla\times\vec{A} = -\frac{1}{\mu_0\lambda_L^2}\vec{B}.$$

And from Maxwell's equations, we know that

$$\begin{aligned}\nabla\times\vec{B} &= \mu_0\vec{j}\\ \nabla\times\nabla\times\vec{B} &= -\nabla^2\vec{B} = \mu_0\nabla\times\vec{j}\\ \Rightarrow\quad \nabla^2\vec{B} &= \frac{\vec{B}}{\lambda_L^2}\end{aligned}$$

This equation is called the London equation and as we can see that it implies that $\vec{B}$ cannot be uniform in space, that is, $\vec{B}(x)$ cannot be a constant except when $\vec{B}=0$. And for $\vec{B}=0$, $\vec{j}$ must be zero, which is different from the case of the perfect conductor. The general solution for this equation is

$$B(x) = B_0 e^{-x/\lambda_L},$$

Later in this chapter, we will derive the expression for the London penetration depth. However, for now we will simply state the result of that derivation

$$\lambda_L = \left(\frac{\varepsilon_0 mc^2}{nq^2}\right)^{1/2},$$

where m and q are a mass and charge of the particle and n is its concentration. The London penetration depth indicates the length scale from the surface which superconductors would allow magnetic flux to penetrate. Normally, superconductivity disappears in the penetrated region.

Coherence Length The coherence length ξ is the length scale of the layer connecting the normal state, which the magnetic flux can reside, and the superconducting state where the magnetic flux is absent. The difference between ξ and λ_L can be used to characterize the type of the superconductors. For Type-I superconductor, $\xi \gg \lambda$. On the other hand, for Type-II superconductor, $\xi \ll \lambda$.

We can approximate the coherence length from the energy gap using the Heisenberg uncertainty principle, that is,

$$\Delta = \delta E = \delta\left(\frac{P^2}{2m}\right) = \frac{P_F}{m}\delta P = v_F\delta P,$$

where P_F and v_F are the Fermi momentum and Fermi velocity. From the uncertainty principle, we have

$$\xi_0 \sim \frac{h}{\delta P} \sim \frac{hv_F}{\Delta} \sim \frac{1}{k_F}\frac{E_F}{\Delta}.$$

Normally, $E_F \sim 10^3 - 10^4\Delta$ and $k_F \sim 10^8\quad \mathrm{m}^{-1}$. Therefore, the coherence length comes out to be about 10^3 Å, which is about 10^3 times of the lattice constant.

10.3 Microscopic Theory

In this section, we will attempt to calculate the energy gap from the microscopic theory. This microscopic theory is called the **BCS theory** after three physicists Bardeen, Cooper, and Shreiffer who published their much celebrated paper on superconductivity in 1957. The BCS theory of superconductivity successfully explain the phenomenon and has the following key features.

1. The attraction between electrons creates a lower ground state compared to the free electron model and this ground state is separated from the lowest excited state by an energy gap.
2. The attractive interaction between electrons is due to lattice vibrations or phonons.
3. The transition temperature T_c can be calculated from the density of states of one electron at the Fermi level and the electron lattice interaction.
4. Thermodynamic properties and the critical field can be explained in terms of the energy gap while the penetration depth and the coherence length are the results of the BCS theory.

Let us consider a two-electron state, which can be written as

$$\psi(\vec{r}_1,\vec{r}_2)=\sum_{\vec{k}} g(\vec{k})e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)},$$

where $g(\vec{k})=0$ if $|\vec{k}|<k_F$. The Hamiltonian can be written as

$$\mathscr{H}\psi=-\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2\right)\psi+V\left(\vec{r}_1,\vec{r}_2\right)\psi=\left(E+E_F\right)\psi.$$

$V\left(\vec{r}_1,\vec{r}_2\right)$, which is due to the electron-phonon interaction, gives rise to the binding energy between two electrons with opposite spins, where the magnitude of the momenta of the two electrons are equal but their direction are opposite. In this case, $E<0$ since the energy of the system decreases by promoting some electrons outside the Fermi sea and binding them. By substituting the wave function to the Hamiltonian, we obtain

$$\frac{\hbar^2k^2}{m}g(\vec{k})+\sum_{k'}g(\vec{k}')V_{kk'}=\left(E+2E_F\right)g(\vec{k}),$$

where we cancel out the $\sum_k e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}$, and $V_{kk'}$ is a Fourier transform of $V\left(\vec{r}_1,\vec{r}_2\right)$, that is,

$$V_{kk'}=\frac{1}{L^3}\int V\left(\vec{r}\right)\,e^{-i\vec{r}\cdot\left(\vec{k}-\vec{k}'\right)}\,d\vec{r}$$

We can approximate that the integral yields a constant and write $V_{kk'}$ as

$$V_{kk'}=\left\{\begin{array}{ll}-\frac{V}{L^3}, & \text{if } \frac{\hbar^2k2}{2m}<E_F+\hbar\omega_D\\ 0, & \text{otherwise}\end{array}\right\},$$

where ω_D is the Debye frequency, which implies the range of energy within which phonons can bind two electrons. Substituting the expression for $V_{kk'}$ to the equation above, we get

$$\begin{aligned}1 &= \frac{V}{L^3}\sum_k\frac{1}{\frac{\hbar^2k^2}{m}-E-2E_F}\\ &= V\int_0^{\hbar\omega_D}D(\varepsilon)\frac{1}{2\varepsilon-E}\,d\varepsilon,\end{aligned}$$

where $2\varepsilon=\frac{\hbar^2k^2}{m}-2E_F$. Normally, $\hbar\omega_D\ll E_F$. Therefore, we can make the approximation that $D(\varepsilon)=D(0)$, which is the density of states at the Fermi energy.

$$\begin{aligned}1 &= D(0)V\int_0^{\hbar\omega_D}\frac{1}{2\varepsilon-E}\,d\varepsilon\\ &= D(0)V\frac{1}{2}\log\left(2\varepsilon-E\right)\Big|_0^{\hbar\omega_D}\\ &= D(0)V\frac{1}{2}\log\left(\frac{2\hbar\omega_D-E}{-E}\right)\\ \Rightarrow\quad \frac{E-2\hbar\omega_D}{E} &= e^{\frac{2}{D(0)V}},\end{aligned}$$

but we know that $D(0)V \ll 1$, that is, $e^{\frac{2}{D(0)V}} \gg 1$

$$\Rightarrow \quad -\frac{2\hbar\omega_D}{E} = e^{\frac{2}{D(0)V}}. \Rightarrow \quad E = -2\hbar\omega_D e^{-\frac{2}{D(0)V}} = -\Delta,$$

which is the expression for the energy gap at $T = 0$. As we can see that the energy gap depends on three parameter, namely ω_D, which indicates the strength of phonons, $D(0)$ which indicates the number of electrons available for binding, and V which indicates the strength of the interaction between two electrons or the **Cooper paris**. For the expression for the temperature dependence of the energy gap, which we will not derive here, we have

$$\frac{\Delta(T)}{\Delta(0)} = 1.74\left[1 - \frac{T}{T_C}\right]^{1/2},$$

where $kT_C = \frac{\Delta(0)}{1.764}$. Similarly, we can derive the expression for the critical field and the specific heat as a function of temperature from the microscopic theory. The results are

$$\frac{H_C(T)}{H_C(0)} = 1 - \left(\frac{T}{T_C}\right)^2,$$

for the critical field and

$$\frac{C_S}{\gamma T_C} = 1.34\left(\frac{\Delta(0)}{T}\right)^{3/2} e^{-\frac{\Delta(0)}{T}},$$

for the specific heat. The jump in the specific heat at T_C is equal to

$$\left.\frac{C_S - C_n}{C_n}\right| = 1.43,$$

where C_n is the specific heat of the normal state.

The BCS theory is very successful in describing physics of conventional superconductors. However, it fails to explain superconductivity in the cuprates, high-T_C superconductors with the copper oxide planes. One thing we know is that the binding energy of the Cooper pair cannot come from phonons, since it would give rise to very low T_C. However, no one know exactly what mehanism give rise to T as high as 150 K. Currently, physicists around the world are intensively looking for clues to explain high-T_C superconductivity and many believe that its many secrets lie in the magnetism, which we will study in the next chapter.

10.4 Josephson tunneling

In this section, we will consider the tunneling through a junction of a thin insulator. First, let us consider two pieces of conductors separated by a thin layer of a insulator. Normally, the insulating layer will not allow a current through. However, if the layer is thin enough ($10 - 20$ Å), then it is possible for electrons to go though the potential barrier, and we call this phenomenon **tunneling**. The $I - V$ curve for the junction with normal conductors is of the Ohmic type, that is, for low voltage the current is linearly proportional to the applied voltage.

However, if on side of the junction is a normal metal while the other side is a superconductor, then at $T \to 0$ there will be no current until the voltage reaches some finite value, which is equal to $\frac{\Delta}{2e}$ where Δ is the energy of the superconducting gap. If T is small but not equal to zero, then there could be a small current leaking through the junction. This small current is due to the thermally excited electrons, which no longer form Cooper paris. On the normal conductor side, the electrical transport is due to single electrons, whereas on the superconducting side, the transport is dominated by Cooper pairs, which is superconducting. Therefore, when electrons tunnel through the junction from the superconducting side to the normal metal side the pairing has to be broken, which requires the energy of $\frac{\Delta}{2e}$ per one electron; note that we have to divide by two since there are two electrons for one Cooper pair.

Now, for the Josephson junction, the two sides of the junction are both superconductors which could be of the same kind or of the different kinds. For simplicity, we will only consider the case where both sides are the same type of the superconductor. Let us consider what we can expect if the two sides are both superconductors. In this case, the Cooper pairs do not need to break into single electrons in order to conduct on the other side. And since the Cooper pairs can tunnel through the junction if the junction is thin enough, there can be current through the junction even without voltage across the junction. It turns out that there are two types of the Josephson effect:

1. **DC Josephson effect**: The direct current can occur without applied electric or magnetic field.

2. **AC Josephson effect**: When the dc voltage is applied across the junction, the rf current is induced. This phenomenon is used to measure the value of $\hbar/e$ with high accuracy.

DC Josephson effect First, we will try to explain the DC Josephson effect. Let ψ_1 and ψ_2 be the probability amplitudes of electron pairs on two sides of the junction, where 1 denotes the left hand side and 2 denotes the right hand side. Now, let us consider the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \mathscr{H}\psi,$$

where $\mathscr{H} = \hbar T$ is the tunneling operator through the insulating junction. Therefore, for ψ_1 and ψ_2, we obtain

$$\begin{aligned} i\hbar\frac{\partial\psi_1}{\partial t} &= \hbar T\psi_2 \\ i\hbar\frac{\partial\psi_2}{\partial t} &= \hbar T\psi_1, \end{aligned}$$

that is, the change in ψ_1 (ψ_2) is caused by the tunneling from ψ_2 (ψ_1). Note that the unit of T is frequency or we can think of T as the rate of tunneling, and if $T = 0$, then there is no tunneling. It is a measure of how much ψ_1 can 'leak' to ψ_2 and vice versa. For the wave functions, we will let

$$\begin{aligned} \psi_1 &= n_1^{1/2}e^{i\theta_1} \\ \psi_2 &= n_2^{1/2}e^{i\theta_2}, \end{aligned}$$

where n_1 and n_2 are the density of Cooper pairs on sides 1 and 2. Now we can substitute the wave functions into the Schrödinger equation and obtain

$$\begin{aligned} \frac{\partial\psi_1}{\partial t} &= \frac{1}{2}n_1^{-1/2}e^{i\theta_1}\frac{\partial n_1}{\partial t} + i\psi_1\frac{\partial\theta_1}{\partial t} = -iT\psi_2 \\ \frac{\partial\psi_2}{\partial t} &= \frac{1}{2}n_2^{-1/2}e^{i\theta_2}\frac{\partial n_2}{\partial t} + i\psi_2\frac{\partial\theta_2}{\partial t} = -iT\psi_1. \end{aligned}$$

We multiply the first equation by $n_1^{1/2}e^{-i\theta_1}$ and the second equation by $n_2^{1/2}e^{-i\theta_1}$.

$$\begin{aligned} \frac{1}{2}\frac{\partial n_1}{\partial t} + in_1\frac{\partial\theta_1}{\partial t} &= -iT\left(n_1n_2\right)^{1/2}e^{i\delta} \\ \frac{1}{2}\frac{\partial n_2}{\partial t} + in_2\frac{\partial\theta_2}{\partial t} &= -iT\left(n_1n_2\right)^{1/2}e^{-i\delta}, \end{aligned}$$

where $\delta = \theta_2 - \theta_1$. We will solve this set of equations by equating the real part and the imaginary part.

$$\begin{aligned}
\frac{\partial n_1}{\partial t} &= 2T\,(n_1 n_2)^{1/2} \sin\delta \\
\frac{\partial n_2}{\partial t} &= -2T\,(n_1 n_2)^{1/2} \sin\delta \\
\frac{\partial \theta_1}{\partial t} &= -T\left(\frac{n_2}{n_1}\right)^{1/2} \cos\delta \\
\frac{\partial \theta_2}{\partial t} &= -T\left(\frac{n_1}{n_2}\right)^{1/2} \cos\delta.
\end{aligned}$$

Since we assume that the superconductors on both side of the junction are of the same type, n_1 must be approximately the same as n_2, that is, $n_1 \approx n_2$.

$$\Rightarrow \quad \frac{\partial \theta_1}{\partial t} \approx \frac{\partial \theta_2}{\partial t}; \quad \frac{\partial}{\partial t}(\theta_2 - \theta_1) = 0$$

and the current becomes

$$\frac{\partial n_2}{\partial t} = -\frac{\partial n_1}{\partial t} \equiv J,$$

which is the current flow from one side of the junction to the other side. The expression for J is given by

$$J = 2T\,(n_1 n_2)^{1/2} \sin\delta \equiv J_0 \sin\delta.$$

We can see that J is not zero as long as the phase difference δ is not zero, and that there can be current even though there is no voltage across the junction.

AC Josephson effect Next we will apply a DC voltage across the junction. The electron pairs now have the potential energy equal to $qV = -2eV$. This is the voltage difference across the junction, and we can think of it as the potential on the left hand side (1) is equal to $-eV$ and that on the right hand side (2) is equal to eV. Our Schrödinger equations, hence, become

$$\begin{aligned}
i\hbar \frac{\partial \psi_1}{\partial t} &= \hbar\psi_2 - eV\psi_1 \\
i\hbar \frac{\partial \psi_2}{\partial t} &= \hbar\psi_1 + eV\psi_2.
\end{aligned}$$

Most of our calculations will be similar to the first case except that now we have an additional term of the potential energy. Substituting the wave function gives us

$$\begin{aligned}
\frac{1}{2}\frac{\partial n_1}{\partial t} + i n_1 \frac{\partial \theta_1}{\partial t} &= \frac{ieVn_1}{\hbar} - iT\,(n_1 n_2)^{1/2} e^{i\delta} \\
\frac{1}{2}\frac{\partial n_2}{\partial t} + i n_2 \frac{\partial \theta_2}{\partial t} &= -\frac{ieVn_2}{\hbar} - iT\,(n_1 n_2)^{1/2} e^{-i\delta},
\end{aligned}$$

The real and the imaginary parts can be written as

$$\begin{aligned}
\frac{\partial n_1}{\partial t} &= 2T\,(n_1 n_2)^{1/2} \sin\delta \\
\frac{\partial n_2}{\partial t} &= -2T\,(n_1 n_2)^{1/2} \sin\delta \\
\frac{\partial \theta_1}{\partial t} &= \frac{eV}{\hbar} - T\left(\frac{n_2}{n_1}\right)^{1/2} \cos\delta \\
\frac{\partial \theta_2}{\partial t} &= -\frac{eV}{\hbar} - T\left(\frac{n_1}{n_2}\right)^{1/2} \cos\delta.
\end{aligned}$$

As you can see the real part remains the same as in the first case, but the imaginary part now contains an extra term, namely $\frac{eV}{\hbar}$. This term enables us to measure the ratio of $e/\hbar$ to a very high degree of accuracy. Again, we will assume that the two superconductors on both sides of the junction are the same, and hence $n_1 \approx n_2$. Therefore,

$$\begin{aligned} \frac{\partial}{\partial t}(\theta_2 - \theta_1) &= \frac{\partial \delta}{\partial t} = -\frac{2eV}{\hbar} \\ \Rightarrow \quad \delta(t) &= \delta(0) - \frac{2eV}{\hbar}t. \end{aligned}$$

And the current becomes

$$J = J_0 \sin\left[\delta(0) - \frac{2eV}{\hbar}t\right].$$

As you can see the current now oscillates as a function of time with an angular frequency $\omega = \frac{2eV}{\hbar}$ since the phase difference is now changing with time. By measuring this frequency, we can obtain the value of $\frac{e}{\hbar}$.

Macroscopic Quantum Interference Now we will try to make a device out of the Josephson junction by connecting two junctions and make them into a loop. This device can be used to measure magnetization quite accurately. This idea of this device starts from the quantization of magnetic flux, that is,

$$\begin{aligned} \hbar c \oint \nabla\theta \cdot d\vec{l} &= q \oint \vec{A} \cdot d\vec{l} \\ \hbar c \cdot 2\pi n &= q \int \nabla \times \vec{A} \cdot d\vec{\sigma} = q \int \vec{B} \cdot d\vec{\sigma} = q\Phi \\ \Rightarrow \quad \Phi = \frac{2\pi\hbar c}{q} n, & \end{aligned} \tag{10.1}$$

where n is an integer. However, if the integral does not cover the whole loop then the phase difference becomes

$$\theta_2 - \theta_1 = \frac{q\Phi}{\hbar c} = \frac{2e\Phi}{\hbar c},$$

where $|q| = 2e$ for the Cooper pair. Therefore, the total phase difference of each junction δ_a for the top loop and δ_b for the bottom loop is given by

$$\delta_b = \delta_0 + \frac{e}{\hbar c}\Phi \quad \text{and} \quad \delta_a = \delta_0 - \frac{e}{\hbar c}\Phi,$$

and the expression for the total current is given by

$$\begin{aligned} J_{total} &= J_0\left[\sin\left(\delta_0 + \frac{e}{\hbar c}\Phi\right) + \sin\left(\delta_0 - \frac{e}{\hbar c}\Phi\right)\right] \\ &= 2J_0 \sin\delta_0 \cos\frac{e\Phi}{\hbar c}. \end{aligned} \tag{10.2}$$

Again, we can see that if we vary the magnetic flux Φ then the current changes between the maximum value of J_0 to the minimum value of $-J_0$. By measuring this oscillation as we move the magnetic material through the device, we can measure the magnetization of that material. The device that uses this principle to measure magnetization is called SQUID, which stands for **S**uperconducting **QU**antum **I**nterfernce **D**evice.

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