IDENTITIES INVOLVING THE TRIBONACCI NUMBERS SQUARED VIA TILING WITH COMBS

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Abstract. The number of ways to tile an n-board (an $n\times 1$ rectangular board) with $(\frac{1}{2},\frac{1}{2};1)$-, $(\frac{1}{2},\frac{1}{2};2)$-, and $(\frac{1}{2},\frac{1}{2};3)$-combs is T_{n+2}^2 where T_n is the nth tribonacci number. A $(\frac{1}{2},\frac{1}{2};m)$-comb is a tile composed of m sub-tiles of dimensions $\frac{1}{2}\times 1$ (with the shorter sides always horizontal) separated by gaps of dimensions $\frac{1}{2}\times 1$. We use such tilings to obtain quick combinatorial proofs of identities relating the tribonacci numbers squared to one another, to other combinations of tribonacci numbers, and to the Fibonacci, Narayana's cows, and Padovan numbers. Most of these identities appear to be new.

1. Introduction

It is well-known that a combinatorial interpretation of the Fibonacci number F_{n+1} (where $F_n = F_{n-1} + F_{n-2}$, $F_1 = 1$, $F_0 = 0$) is the number of ways to tile an n-board (a linear array of n unit square cells) with tiles of size 1×1 and 2×1 [3]. As a generalization of this, combs were recently introduced to give a combinatorial interpretation of $s_n^r s_{n+1}^{p-r}$ where $s_n = \sum_{i=1}^{q} v_i s_{n-m_i}$, $s_0 = 1$, $s_{n<0} = 0$, where p, v_i, and m_i are positive integers, $m_1 < \cdots < m_q$, and $r = 0, \ldots, p-1$ [1]. A $(w, g; m)$-comb is a tile formed from a linear array of m sub-tiles (we call *teeth*) each of size $w \times 1$ (with the sides of length w aligned horizontally) and separated from one another by gaps of size $g \times 1$. A $(w, g; 1)$-comb is just a $w \times 1$ rectangular tile. A $(w, g; 2)$-comb is also known as a (w, g)-fence. Fences were first introduced to provide a combinatorial interpretation of the tribonacci numbers (which are defined by $T_n = T_{n-1} + T_{n-2} + T_{n-3} + \delta_{n,2}$, $T_{n<2} = 0$, where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise) via tiling an n-board using just two types of tile, namely, squares and $(\frac{1}{2}, 1)$-fences [4]. Since then, we have used the tiling of n-boards with various types of fence and rectangular or square tiles to obtain combinatorial proofs of identities involving the Fibonacci numbers [6, 8, 7, 9].

To obtain a combinatorial interpretation of the squares of various number sequences via the tiling of an n-board with combs, we require the following theorem which is a special case of Corollary 2.2 of [1].

Theorem 1.1. *The number of ways to tile an n-board using $(\frac{1}{2},\frac{1}{2};m_i)$-combs for $i = 1,\ldots,q$ with $0 < m_1 < \cdots < m_q$ is s_n^2 where $s_n = s_{n-m_1} + \cdots + s_{n-m_q} + \delta_{n,0}$.*

For a combinatorial interpretation of the tribonacci numbers squared we therefore need to consider tiling an n-board with $(\frac{1}{2},\frac{1}{2};1)$-, $(\frac{1}{2},\frac{1}{2};2)$-, and $(\frac{1}{2},\frac{1}{2};3)$-combs, which from now on we will refer to as half-squares (h), fences (f), and combs (c), respectively. This is formalized in the following theorem (which is a particular instance of Theorem 1.1).

Theorem 1.2. *Let A_n be the number of ways to tile an n-board using half-squares, fences, and combs. Then $A_n = T_{n+2}^2$.*

The tiling of a board using tiles with gaps (or tiles with any sides which are not of integer length) can be treated as a tiling using metatiles. A metatile is a grouping of tiles that exactly covers an integer number of cells without any gaps and cannot be split into smaller metatiles [4]. In some cases, there is a finite set of possible metatiles; e.g., when tiling with squares and $(\frac{1}{2}, 1)$-fences, the metatiles are a square, a fence with its gap filled by a square, and three interlocking fences, and these are of lengths 1, 2, and 3, respectively [4]. Interesting identities are obtained when there are metatiles of arbitrary length if an expression for (or at least a recursion relation giving) the number of metatiles of a given length can be obtained [6, 7, 1]. Obtaining such an expression has been achieved via a digraph approach [5], determining the form of all possible metatiles [6], finding a bijection between the metatiles and some other objects whose number is known [1], and by examining how all possible metatiles of length l can be obtained by replacing tiles at the ends of metatiles of length $l-1$ [7]. It is this final approach that we will use in the next section. In Section 3 we use the expressions for the number of metatiles to obtain identities using similar methods to [7].

2. Metatiles

As in [7, 1], we define a *mixed metatile* as a metatile that contains more than one type of tile. Thus all the metatiles are mixed except for h^2, f^2 (the bifence), and c^2 (the bicomb) which are shown in Fig. 1. We refer to each half of each cell on the n-board as a *slot* [7, 1]. The following lemma and its proof are analogous to Lemma 2.3 in [1].

Lemma 2.1. *Each mixed metatile is a member of a pair of mixed metatiles. One member of the pair is obtained from the other by swapping the contents of the slots in each cell.*

Proof. The swapping operation will only generate the same metatile if the teeth in each pair of slots are from the same type of tile. This can only occur if the metatile is not mixed. The swapping operation will not split a metatile into more than one metatile since the operation does not change which cells any given fence or comb straddles. □

We use a 2-digit string σ to specify the contents of the final 2 slots of a metatile. A digit m in the string indicates that the slot is filled by a tooth belonging to a $(\frac{1}{2}, \frac{1}{2}; m)$-comb. Let $\mu_l^{[\sigma]}$ be the number of metatiles of length l that end with σ. Then by Lemma 2.1,

$$\mu_l^{[m_1 m_2]} = \mu_l^{[m_2 m_1]}, \tag{2.1}$$

where the m_i are the digits of σ. As an example, hfh and fh^2 (Fig. 1) are a pair of mixed metatiles in the sense of Lemma 2.1.

The digits in σ for a mixed metatile must be distinct and, in view of (2.1), we need only consider $\mu_l^{[12]}$, $\mu_l^{[13]}$, and $\mu_l^{[23]}$.

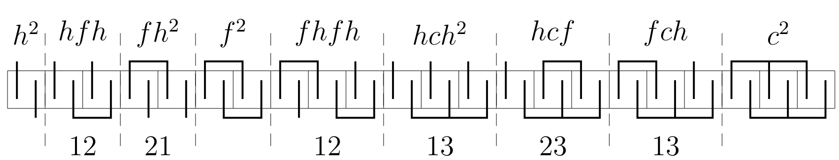

Figure 1. A 22-board tiled with metatiles of length $l \leq 3$. Each bold vertical line represents a $(\frac{1}{2} \times 1)$ tooth and is aligned with the middle of the tooth it represents. Bold horizontal lines indicate which teeth are part of the same fence or comb. In the case of the $l = 3$ mixed metatiles, only one element of each pair is shown. Dashed lines show boundaries between metatiles. The symbolic representation is above each metatile. σ for the mixed metatiles appears below.

Lemma 2.2. *For all integers l, when tiling with h, f, and c,*

$$\mu_l^{[12]} = \mu_{l-1}^{[12]} + \mu_{l-2}^{[12]} + \mu_{l-3}^{[12]} + \delta_{l,2} + \delta_{l,4}, \quad \mu_{l<2}^{[12]} = 0, \tag{2.2a}$$
$$\mu_l^{[13]} = \mu_{l-1}^{[13]} + \mu_{l-2}^{[13]} + \mu_{l-3}^{[13]} + 2\delta_{l,3}, \quad \mu_{l<3}^{[13]} = 0, \tag{2.2b}$$
$$\mu_l^{[23]} = \mu_{l-1}^{[23]} + \mu_{l-2}^{[23]} + \mu_{l-3}^{[23]} + \delta_{l,3} + \delta_{l,5}, \quad \mu_{l<3}^{[23]} = 0. \tag{2.2c}$$

Proof. To form metatiles of length l from a given metatile of length $l-1$ we replace an h in the final slot by an f, and/or replace an f by a c. A remaining empty slot in the lth cell is filled with an h. A metatile with $\sigma = 12$ (e.g., hfh) will thus generate metatiles with $\sigma = 21$ (by replacing the final h by an f and adding an h, e.g., hf^2h which is paired with $fhfh$), $\sigma = 13$ (by replacing the final f by a c and adding an h in the gap, e.g., hch^2), and $\sigma = 23$ (by replacing both, e.g., hcf). Similarly, metatiles with $\sigma = 13$ and $\sigma = 23$ will generate metatiles with $\sigma = 21$ and $\sigma = 31$, respectively. Hence, after using (2.1),

$$\mu_l^{[12]} = \mu_{l-1}^{[12]} + \mu_{l-1}^{[13]} + \delta_{l,2}, \tag{2.3a}$$
$$\mu_l^{[13]} = \mu_{l-1}^{[12]} + \mu_{l-1}^{[23]} + \delta_{l,3}, \tag{2.3b}$$
$$\mu_l^{[23]} = \mu_{l-1}^{[12]}, \tag{2.3c}$$

where the $\delta_{l,2}$ and $\delta_{l,3}$ terms arise from the creation of metatiles hfh and fch from the non-mixed metatiles h^2 and f^2, respectively. Substituting (2.3b) into (2.3a) and then (2.3c) into the result gives (2.2a). Then (2.2c) immediately follows from applying (2.2a) to (2.3c). Substituting (2.2a) and (2.2c) into (2.3b) and regrouping terms gives (2.2b). □

Let μ_l be the number of mixed metatiles of length l. Then $\mu_l = 2(\mu_l^{[12]} + \mu_l^{[13]} + \mu_l^{[23]})$ and so from summing (2.2) we obtain

$$\mu_l = \mu_{l-1} + \mu_{l-2} + \mu_{l-3} + 6\delta_{l,3} + 2(\delta_{l,2} + \delta_{l,4} + \delta_{l,5}), \quad \mu_{l<2} = 0. \tag{2.4}$$

This gives $\{\mu_l\}_{l\ge 2} = 2, 8, 12, 24, 44, 80, 148, 272, 500, \ldots$, which after a few terms is one form of tribonacci sequence. However, μ_l can be expressed in terms of the usual tribonacci numbers as shown in the following lemma.

Lemma 2.3. *The number of mixed metatiles of length l when tiling with h, f, and c is*

$$\mu_l = 4(T_l + T_{l-1}) - 2\delta_{l,2}.$$

Proof. The generating function for (2.4) is $(2z^2 + 6z^3 + 2z^4 + 2z^5)/(1 - z - z^2 - z^3)$ which can be re-expressed as $4(z^2 + z^3)/(1 - z - z^2 - z^3) - 2z^2$. As the generating function for T_l is $z^2/(1 - z - z^2 - z^3)$, the result follows. □

3. Identities

Lemma 3.1. *For all non-negative integers n,*

$$A_n = \delta_{n,0} + A_{n-1} + 3A_{n-2} + 9A_{n-3} + \sum_{l=4}^{n} \mu_l A_{n-l}, \tag{3.1}$$

where $A_n = 0$ for $n < 0$.

Proof. Following [2, 5, 6], we condition on the last metatile. If the last metatile is of length l there will be A_{n-l} ways to tile the remaining $n - l$ cells. There is one metatile of length 1 (h^2), three of length 2, nine of length 3, and μ_l metatiles of length l for each $l \ge 4$. If $n = l$ there is exactly one tiling corresponding to that final metatile so we make $A_0 = 1$. There is no way to tile an n-board if $n < l$ and so $A_{n<0} = 0$. □

Identity 3.2. *For all non-negative integers n,*

$$T_n^2 = \delta_{n,2} + T_{n-1}^2 + 3T_{n-2}^2 + 9T_{n-3}^2 + 4\sum_{l=4}^{n-2}(T_l + T_{l-1})T_{n-l}^2.$$

Proof. It follows from Lemma 3.1, Lemma 2.3, and Theorem 1.2. □

Identity 3.3. *For $n \geq 0$, $T_n^2 = \delta_{n,2} - \delta_{n,3} - \delta_{n,4} - \delta_{n,5} + 2T_{n-1}^2 + 3T_{n-2}^2 + 6T_{n-3}^2 - T_{n-4}^2 - T_{n-6}^2$.*

Proof. Letting $E(n)$ represent (3.1) and re-indexing three of the sums in $E(n) - E(n-1) - E(n-2) - E(n-3)$ gives

$$A_n = \delta_{n,0} - \delta_{n,1} - \delta_{n,2} - \delta_{n,3} + 2A_{n-1} + 3A_{n-2} + 6A_{n-3} - A_{n-4} - A_{n-6} + \sum_{l=7}^{n}(\mu_l - \mu_{l-1} - \mu_{l-2} - \mu_{l-3})A_{n-l}.$$

after using $\mu_4 = 12$, $\mu_5 = 24$, and $\mu_6 = 44$. The sum vanishes by virtue of (2.4) and, after changing n to $n-2$, the identity follows from Theorem 1.2. □

Identity 3.4. *For all non-negative integers n,*

$$T_{n+4}^2 = 1 + \sum_{k=0}^{n}\left\{3T_{k+2}^2 + 9T_{k+1}^2 + 4\sum_{i=2}^{k}(T_{k+4-i} + T_{k+3-i})T_i^2\right\}.$$

Proof. How many ways are there to tile an $(n+2)$-board using at least one fence or comb? *Answer 1*: $A_{n+2} - 1$ since this corresponds to all tilings except the all-h tiling. *Answer 2*: condition on the location of the last metatile containing an f or c. If the metatile is of length l and ends on cell $k+2$ (for $k = l-2, \ldots, n$) then the number of ways to tile the board is $(\mu_l + \delta_{l,2} + \delta_{l,3})A_{k+2-l}$. Summing over all possible lengths and all possible k, introducing $i = k+2-l$ and summing over that rather than l, and then equating to Answer 1 gives

$$A_{n+2} - 1 = \sum_{k=0}^{n}\sum_{i=0}^{k}(\mu_{k+2-i} + \delta_{i,k} + \delta_{i,k-1})A_i.$$

Taking the $i = k$ and $i = k-1$ terms outside the sum over i, changing i to $i-2$, and then using Theorem 1.2 and Lemma 2.3 gives the identity. □

Identity 3.5. *For all non-negative integers n and $j = 0, 1$,*

$$T_{2(n+1)+j}^2 = 1 + \sum_{k=1}^{n}\left\{T_{2k+j+1}^2 + 2T_{2k+j}^2 + 9T_{2k+j-1}^2 + 4\sum_{i=0}^{2k+j-4}(T_{2k+j-i} + T_{2k+j-i-1})T_{i+2}^2\right\}.$$

Proof. How many ways are there to tile an $(2n+j)$-board using at least one half-square or comb? *Answer 1*: $A_{2n+j} - \delta_{0,j}$ since only the all-bifence tiling has no h or c and this only occurs for even length boards. *Answer 2*: condition on the location of the last metatile containing an h or c. The last cell of this metatile must lie on cell $2k+j$ for some $k = \delta_{0,j}, \ldots, n$ since the cells to the right must be filled with bifences. If the metatile is of length l then the number of ways to tile the board is $(\mu_l + \delta_{l,1} + \delta_{l,3})A_{2k+j-l}$. Summing over all possible l and k, separating out the $l = 1$ case, and then equating to Answer 1 gives

$$A_{2n+j} - \delta_{j,0} = \sum_{k=\delta_{j,0}}^{n} A_{2k+j-1} + \sum_{k=1}^{n}\sum_{i=0}^{2k+j-2}(\mu_{2k+j-i} + \delta_{3,2k+j-i})A_i.$$

Simplifying and then using Theorem 1.2 and Lemma 2.3 gives the identity. □

Identity 3.6. *For all non-negative integers n and $j = 0, 1, 2$,*

$$T_{3n+2+j}^2 = 1 + 3\delta_{j,2} + \sum_{k=1}^{n} \left\{ T_{3k+j+1}^2 + 3T_{2k+j}^2 + 4 \sum_{i=0}^{3k+j-3} (T_{3k+j-i} + T_{3k+j-i-1})T_{i+2}^2 \right\}.$$

Proof. How many ways are there to tile an $(2n+j)$-board using at least one half-square or fence? *Answer 1*: $A_{3n+j} - \delta_{0,j}$ since only the all-bicomb tiling has no h or f and this only occurs for boards of length divisible by 3. *Answer 2*: condition on the location of the last metatile containing an h or f. The last cell of this metatile must lie on cell $3k + j$ for some $k = \delta_{0,j}, \ldots, n$ since the cells to the right must be filled with bicombs. If the metatile is of length l then the number of ways of tile the board is $(\mu_l + \delta_{l,1} + \delta_{l,2})A_{3k+j-l}$. Summing over all possible l and k, separating out the $l = 1$ and $l = 2$ cases, equating the whole expression to Answer 1, simplifying, and then using Theorem 1.2 and Lemma 2.3 gives the identity. □

The number of ways to tile an n-board using only h^2, f^2, and c^2 is T_{n+2} since these metatiles are of lengths 1, 2, and 3.

Identity 3.7. *For all non-negative integers n,*

$$T_n^2 = T_n + \sum_{k=2}^{n-2} \sum_{l=2}^{k} \{4(T_l + T_{l-1}) - 2\delta_{l,2}\} T_{k-l+2}^2 T_{n-k}.$$

Proof. How many ways are there to tile an $(n-2)$-board using at least one mixed metatile? *Answer 1*: $A_{n-2} - T_n$ since T_n is the number of ways to tile an $(n-2)$-board without using mixed metatiles. *Answer 2*: condition on the last mixed metatile. If this metatile is of length l and ends on cell k where $k = l, \ldots, n-2$, then the number of ways to tile the remaining cells is $A_{k-l}T_{n-2-k}$. Given that there are μ_l mixed metatiles of length l, summing over all possible k and l and equating to Answer 1 gives

$$A_{n-2} - T_n = \sum_{k=2}^{n-2} \sum_{l=2}^{k} \mu_l A_{k-l} T_{n-2-k}.$$

Using Theorem 1.2 and Lemma 2.3 gives the identity. □

The number of ways to tile an n-board using only h and f is F_{n+1}^2 and there are 2 metatiles of length l containing only h and c for $l \geq 3$ [6].

Identity 3.8. *For all non-negative integers n,*

$$T_{n+2}^2 = F_{n+1}^2 + \sum_{k=3}^{n} \sum_{l=3}^{k} \{4(T_l + T_{l-1}) + \delta_{l,3} - 2\} T_{k-l+2}^2 F_{n-k+1}^2.$$

Proof. How many ways are there to tile an n-board using at least one comb? *Answer 1*: $A_n - F_{n+1}^2$. *Answer 2*: condition on the last metatile containing a comb. The number of such metatiles of length l (where, owing to the size of a comb, $l \geq 3$) is $\mu_l + \delta_{l,3} - 2$. Suppose such a metatile of length l ends on cell k (where $k = l, \ldots, n$). Then the number of ways to tile the rest of the board is $A_{k-l}F_{n-k+1}^2$. Summing over all possible metatiles, l, and k, and equating to Answer 1 gives

$$A_n - F_{n+1}^2 = \sum_{k=3}^{n} \sum_{l=3}^{k} (\mu_l + \delta_{l,3} - 2) A_{k-l} F_{n-k+1}^2.$$

Using Theorem 1.2 and Lemma 2.3 gives the identity. □

The Narayana's cows sequence is defined by $c_n = c_{n-1} + c_{n-3} + \delta_{n,0}$, $c_{n<0} = 0$. From Theorem 1.1, the number of ways to tile an n-board using only h and c is c_n^2. The number of mixed metatiles of length l containing only h and c is $2p_{l-1}$ where $p_n = p_{n-2} + p_{n-3} + \delta_{n,0}$, $p_{n<0} = 0$ [1]. These are the Padovan numbers (offset from sequence A000931 in [10]). From Theorem 1.1, the number of ways to tile an n-board using only f and c is p_n^2. The number of mixed metatiles of length l containing only f and c is $2c_{l-5}$ [1].

Identity 3.9. *For all non-negative integers n,*

$$T_{n+2}^2 = c_n^2 + \sum_{k=2}^{n}\sum_{l=2}^{k}\{4(T_l + T_{l-1}) - \delta_{l,2} - 2p_{l-1}\}T_{k-l+2}^2 c_{n-k}^2.$$

Proof. How many ways are there to tile an n-board using at least one fence? *Answer 1*: $A_n - c_n^2$. *Answer 2*: condition on the last metatile containing a fence. The number of such metatiles of length l is $\mu_l + \delta_{l,2} - 2p_{l-1}$. The proof then proceeds in an analogous way to that of Identity 3.8. □

Identity 3.10. *For all non-negative integers n,*

$$T_{n+2}^2 = p_n^2 + \sum_{k=1}^{n}\sum_{l=1}^{k}\{4(T_l + T_{l-1}) + \delta_{l,1} - 2\delta_{l,2} - 2c_{l-5}\}T_{k-l+2}^2 p_{n-k}^2.$$

Proof. How many ways are there to tile an n-board using at least one half-square? *Answer 1*: $A_n - p_n^2$. *Answer 2*: condition on the last metatile containing a half-square. The number of such metatiles of length l is $\mu_l + \delta_{l,1} - 2c_{l-5}$. The proof then proceeds as in Identity 3.8. □

Lemma 3.11. *The number of ways to tile a board using h, f, and c such that $\sigma = m_1 m_2$ for $m_1, m_2 \in \{1, 2, 3\}$ is $T_{n+2-m_1}T_{n+2-m_2}$.*

Proof. There is a bijection between the tiling of an ordered pair of n-boards with squares, dominoes, and trominoes and the tiling of an n-board with h, f, and c (see the proof of Theorem 2.1 in [1]) whereby all teeth of a $(\frac{1}{2}, \frac{1}{2}; m)$-comb ending at the left (right) slot the kth cell correspond to an m-omino ending in the kth cell of the first (second) of the pair of boards. Thus when there is a tooth belonging to a $(\frac{1}{2}, \frac{1}{2}; m_i)$-comb in the final left (right) slot there are T_{n-m_i+2} ways to tile the rest of the first (second) board ending in an m_i-omino and hence $T_{n+2-m_1}T_{n+2-m_2}$ ways to tile the remaining cells of both boards with ominoes. □

Identity 3.12. *For integers $n \geq 2$,*

$$T_{n+1}T_n = \sum_{l=2}^{n}(T_l + T_{l-2})T_{n-l+2}^2.$$

Proof. How many ways are there to tile an n-board with an h in the final left slot and an f tooth in the final right slot? *Answer 1*: by Lemma 3.11, $T_{n+1}T_n$. *Answer 2*: there are $\mu_l^{[12]}$ metatiles of length l than can end such a board and thus, summing over all possible l, the number of ways to tile the board is $\sum_{l=2}^{n}\mu_l^{[12]}A_{n-l}$. Equating this to Answer 1 and using Theorem 1.2 and the result from (2.2a) that $\mu_l^{[12]} = T_l + T_{l-2}$ then gives the identity. □

Identity 3.13. *For integers $n \geq 3$,*

$$T_{n+1}T_{n-1} = 2\sum_{l=3}^{n}T_{l-1}T_{n-l+2}^2.$$

Proof. How many ways are there to tile an n-board with an h in the final left slot and a c tooth in the final right slot? *Answer 1*: by Lemma 3.11, $T_{n+1}T_{n-1}$. *Answer 2*: there are $\mu_l^{[13]}$ metatiles of length l than can end such a board. From (2.2b), $\mu_l^{[13]} = 2T_l$. The proof then proceeds in an analogous way to that for Identity 3.12. □

4. Discussion

The unusual feature of this particular selection of tile types is that numbers of metatiles can be simply expressed in terms of the sequence generated from the tiling of the whole board. As a result, most of the identities we obtain involve only the tribonacci numbers. Aside from Identity 3.3, we have not been able to locate any of the remaining identities we derive here in the literature.

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