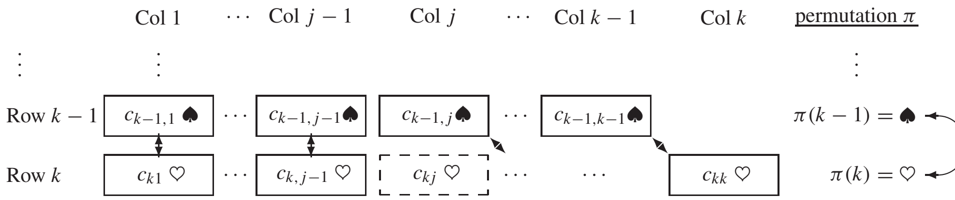

Figure 4. When C_{kj} is the first small card and $j \geq 2$, then swap the first $j-1$ cards of row $k-1$ with the first $j-1$ cards of row k, change the suit of card C_{kj}, then swap the remaining cards of rows $k-1$ and k. In the new Vandermonde table, card C_{kj} remains the first small card.

leaving card C_{kj} in its place, but changing its suit from hearts to spades, then swapping the remaining $k-j$ cards of rows $k-1$ and k, as in Figure 4.

Why is it legal to change the suit of card C_{kj} from hearts to spades? Since C_{kj} was the first small card, then the spade card $C_{k-1,j-1}$ is not small and therefore has a value strictly greater than x_{j-2}. Thus all spade cards can take on values less than or equal to x_{j-1}. Since C_{kj} is small, its value is at most x_{j-1}, so changing it from hearts to spades is allowable.

As before, C_{kj} remains the first small card of C', so $(C')' = C$ and C' has permutation π', which has opposite parity of π since they differ by a transposition. Thus there is a one-to-one correspondence between the positively counted Vandermonde tables with small cards and the negatively counted Vandermonde tables with small cards. Therefore the determinant of V_n is the number of Vandermonde tables with no small cards, namely, $\prod_{0 \leq i < j \leq n}(x_j - x_i)$, as desired.

Remark. For another combinatorial proof of Vandermonde's determinant, where the cancellation occurs in the product instead of the sums, see the short paper by Ira Gessel [**1**].

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Evaluation of Some Improper Integrals Involving Hyperbolic Functions

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In this note I present a result that seems elementary enough to be added to the list of tricks for evaluating integrals taught in a complex variables course, but one to which I have been unable to find any reference. It gives a straightforward procedure that can be used to evaluate a class of integrals some of which do not appear in [**1**] and for which *Mathematica* 5.1 [**2**] generates expressions involving exotic special functions that it cannot simplify further.

Theorem 1. *The principal value of the integral*

$$I_n = \int_{-\infty}^{\infty} x^n f(x)\, dx \qquad (n = 0, 1, 2, \dots) \tag{1}$$

can be evaluated by summing residues of $z^{m+1} f(z)$ for $m = 0, 1, \dots, n$, provided that there exists a positive real number β such that the following conditions are met:

(i) $f(x + i\beta) = f(x)$ *for all* x;

(ii) $(x + i\alpha)^{n+1} f(x + i\alpha) \to 0$ *as* $|x| \to \infty$ *whenever* $0 \le \alpha \le \beta$;

(iii) $f(z)$ *is analytic in an open set that contains* $S = \{z : 0 \le \mathrm{Im}(z) \le \beta\}$ *except at finitely many points* z_j *of* S.

Proof. To find I_n we apply the residue theorem to

$$J_{n+1} \equiv \lim_{R\to\infty} \oint_{\gamma} z^{n+1} f(z)\, dz,$$

where $z = x + iy$ and the contour γ is depicted in Figure 1. Using the properties of $f(x)$ we see that

$$J_{n+1} = I_{n+1} - \int_{-\infty}^{\infty} (x + i\beta)^{n+1} f(x)\, dx,$$

and hence that

$$(n+1) I_n = -\frac{2\pi}{\beta} \sum_j h_j \operatorname{Res}(z^{n+1} f(z), z_j) - \sum_{m=0}^{n-1} \binom{n+1}{m} (i\beta)^{n-m} I_m. \tag{2}$$

Here $\operatorname{Res}(g(z), z_j)$ denotes the residue of $g(z)$ at the singular point $z = z_j$, and h_j is one or one-half depending on whether z_j is inside γ or lies on γ, respectively. When $n = 0$, expression (2) contains only the sum of residues. Using this recurrence relation, I_n can therefore be expressed entirely as sums of residues of $z^{m+1} f(z)$ with $m = 0, \dots, n$. ∎

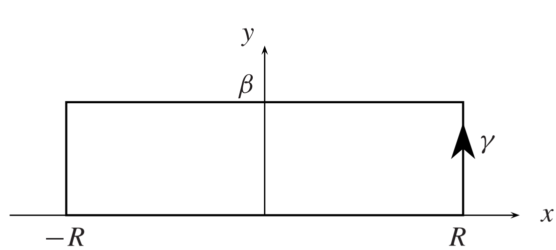

Figure 1. Contour used in the proof of Theorem 1.

Exponential and hyperbolic functions have imaginary periods but not real periods, which means that they exhibit properties (i) and (iii) and can therefore appear as terms in the formula for $f(x)$, provided that $f(x)$ also shows property (ii). We illustrate the method by using it to compute integrals of the form

$$I_{n,m,p} = \int_{-\infty}^{\infty} \frac{x^n \sinh^m x \, dx}{(a + \cosh x)^p} \qquad (a \in \mathbb{R}; \quad n, m, p \in \{0, 1, \dots\}; \quad p > m).$$

Such integrals arise in the stability analysis of solitary wave solutions of certain equations with mixed nonlinearities [**3**]. We first consider $I_{2,0,1}$, which *Mathematica* evaluates in terms of the polylogarithm function and [**1**] gives in a simpler form, but only when $|a| < 1$. In this case $\beta = 2\pi$ and as $f(x)$ is even, $I_1 = 0$. Applying (2) with first $n = 0$ and then $n = 2$ gives

$$I_{2,0,1} = -\frac{1}{3} \sum_j h_j \left\{ 4\pi^2 \operatorname{Res}\left(\frac{z}{a + \cosh z}, z_j \right) + \operatorname{Res}\left(\frac{z^3}{a + \cosh z}, z_j \right) \right\}.$$

When $|a| \neq 1$ the singular points are all simple poles, so the relevant residues are $z_j \operatorname{cosech} z_j$ and $z_j^3 \operatorname{cosech} z_j$, respectively. When $|a| < 1$, the poles are at $(\pi \pm t)i$, where $t = \cos^{-1} a$. This leads to

$$I_{2,0,1} = \tfrac{2}{3}(\pi^2 - t^2) t \operatorname{cosec} t,$$

which is in agreement with result 3 of [**1**, sec. 3.531]. If $a > 1$ the poles are at $\pm\tau + \pi i$, where $\tau = \cosh^{-1} |a|$, and if $a < -1$ the poles are on γ at $\pm\tau$ and $\pm\tau + 2\pi i$. After also obtaining the results when $|a| = 1$, we complete the evaluation of $I_{2,0,1}$ for the remaining values of a:

$$I_{2,0,1} = \begin{cases} \tfrac{2}{3}(\pi^2 + \tau^2)\tau \operatorname{cosech} \tau & \text{if } a \geq 1, \\ \tfrac{2}{3}(2\pi^2 - \tau^2)\tau \operatorname{cosech} \tau & \text{if } a \leq -1, \end{cases}$$

where $\tau \operatorname{cosech} \tau$ is taken to be 1 when $|a| = 1$. In [**1**, sec. 3.533] there is an expression for $I_{n,m,m+1}$ for $m = 1$ but not for larger values of m. For cases where $n \geq 2$ and the integrand is even *Mathematica* gives expressions (involving polylogarithms and the Appell hypergeometric function) that it is unable to simplify. Using our method, we have no difficulty obtaining results in terms of elementary functions for any allowed m, n, and p. For example, we find that if $|a| < 1$,

$$I_{2,2,3} = \tfrac{1}{3} \left\{ \left[6 + (\pi^2 - t^2) \operatorname{cosec}^2 t \right] t - (\pi^2 - 3t^2) \cot t \right\} \operatorname{cosec} t, \qquad t = \cos^{-1} a.$$

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