Growth rate of transverse instabilities of solitary pulse solutions to a family of modified Zakharov-Kuznetsov equations

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Abstract

Using the small-k expansion method, we obtain a closed-form expression for the growth rate of long-wavelength transverse instabilities of solitary pulse solutions to a modified Zakharov-Kuznetsov equation with a nonlinearity of the form $(Au^p + Bu^{2p})u_x$.

Key words: solitons, transverse stability, generalized Zakharov-Kuznetsov equation
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1. Introduction

The Korteweg-de Vries (KdV) equation and its numerous generalizations model weakly nonlinear waves in a wide variety of media (see Ref. [1] for a survey). Here we will be considering one family of such generalizations which takes the form of a modified Zakharov-Kuznetsov (ZK) equation,

$$u_t + (Au^p + Bu^{2p})u_x + C\nabla^2 u_x = 0, \quad (p > 0), \tag{1}$$

where the subscripts denote partial differentiation and $A \neq 0$, B, and $C \neq 0$ are real constants. Equation (1) reduces to the original ZK equation when $p = 1$ and $B = 0$. The ZK equation, first obtained as a description of weakly nonlinear ion-acoustic modes in a strongly magnetized plasma, is of particular interest as it is the simplest equation that admits cylindrical and spherical solitary wave solutions in addition to the more familiar planar KdV soliton solutions [2]. A number of 'modified' ZK equations, most of which can be written in the form of (1), have also been derived [3–7]. To distinguish between them, we refer to a modified ZK equation containing the term $(Au^p + Bu^q)u_x$ as the (p, q)-mZK equation.

ZK-type equations admit planar solitary pulse solutions (independent of y, z) which are stable to perturbations in the direction of propagation provided that $0 < p, q < 4$ [8]. However, these solutions are unstable with respect to long-wavelength transverse perturbations, and the perturbed planar solitary pulses evolve into cylindrical or spherical solitary pulses [9,10]. The growth rate of long-wavelength transverse instabilities has been determined for ZK-type equations with a single nonlinearity for the cases $p = 1, 2, 1/2$ [11–13], and more recently also for the $(1, 2)$- and $(\frac{1}{2}, 1)$-mZK equations [14,15]. In this paper we obtain the corresponding growth rate for the $(p, 2p)$-mZK equation.

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2. Solitary pulse solutions

To simplify the analysis we rescale (1) using

$$u' = \frac{u}{|A|^{1/p}}, \quad t' = \frac{|A|^3 t}{\sqrt{|C|}}, \quad (x', y', z') = \frac{|A|}{\sqrt{|C|}}(x, y, z),$$

which after dropping the primes gives

$$u_t + (\sigma_A u^p + Bu^{2p})u_x + \sigma_C \nabla^2 u_x = 0, \tag{2}$$

where $\sigma_A \equiv \operatorname{sgn}(A)$ and $\sigma_C \equiv \operatorname{sgn}(C)$. This has plane solitary pulse solutions when $\sigma_C B > 0$ or $B = 0$, $\sigma_C = 1$. The solutions obtained in Ref. [16] can be written in the simpler form

$$u(x, t) = \left(\frac{\sigma_A \sigma_C \beta}{\alpha + \cosh \eta(x - Vt)}\right)^{1/p} \tag{3}$$

where η is a free real parameter, $V = \sigma_C \eta^2/p^2$,

$$\beta = |V|(p + 1)(p + 2)\alpha,$$

$$\alpha = \sqrt{\frac{2p + 1}{2p + 1 + BV(p + 1)(p + 2)^2}}.$$

If $\sigma_A \sigma_C = -1$, it can be seen that a further condition for the existence of solitary pulses of the form (3) is that p is rational and has an odd numerator.

Before performing the stability analysis, we can simplify our system further by transforming (2) to a frame moving at speed V along the x-axis and then making the change of variables $u' = \eta^{-2/p}u$, $(t', x', y', z') = \eta(\eta^2 t, x, y, z)$, $B' = \eta^2 B$. Dropping the primes once again and multiplying by σ_C leaves the $(p, 2p)$-mZK equation in the form

$$\sigma_C u_t + \left(\sigma_A \sigma_C u^p + \sigma_C B u^{2p} - \frac{1}{p^2}\right) u_x + \nabla^2 u_x = 0, \tag{4}$$

and after replacing V by σ_C/p^2 in the definitions of α and β, the plane solitary pulse solution reduces to

$$u_0(x) = \left(\frac{\sigma_A \sigma_C \beta}{\alpha + \cosh x}\right)^{1/p}. \tag{5}$$

3. Growth rate of small-k instabilities

To determine the growth rate of long-wavelength transverse instabilities of wavenumber k we use the small-k method [17,1]. We proceed by writing

$$u(x, y, z, t) = u_0(x) + \varepsilon\phi(x)\,\mathrm{e}^{\mathrm{i}ky+\gamma t}, \tag{6}$$

where γ is the growth rate of the perturbation $\varepsilon\phi(x)\,\mathrm{e}^{\mathrm{i}ky}$, and $\varepsilon \ll k^2$. Without loss of generality we have taken the transverse perturbation to be in the y-direction. Substituting (6) into (4) and linearizing with respect to ε, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}L\phi = -\sigma_C \gamma\phi + k^2 \frac{\mathrm{d}\phi}{\mathrm{d}x}, \tag{7}$$

where

$$L \equiv \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sigma_A \sigma_C u_0^p + \sigma_C B u_0^{2p} - \frac{1}{p^2}.$$

We expand ϕ and γ as

$$\phi = \phi_0 + k\phi_1 + k^2\phi_2 + \ldots, \tag{8}$$

$$\gamma = k\gamma_1 + k^2\gamma_2 + \ldots, \tag{9}$$

and substitute (8) and (9) into (7). Equating ascending powers of k in the resulting equation leads to a series of equations that are used to find the ϕ_i and then the γ_i.

At lowest order we have

$$\frac{\mathrm{d}}{\mathrm{d}x}L\phi_0 = 0,$$

which has the solution $\phi_0 = u_{0x}$ by virtue of the x-direction translational invariance of the unperturbed solution. Integrating the equation obtained at first order in k gives

$$L\phi_1 = -\sigma_C \gamma_1 u_0. \tag{10}$$

In previous studies involving ZK-type equations, ϕ_1 is determined by applying the L^{-1} operator [18]. This procedure is somewhat involved as it requires the evaluation of some non-trivial integrals. Here, given the solutions for ϕ_1 obtained for specific ZK-type equations elsewhere, we instead assume that ϕ_1 can be written as a linear sum of xu_{0x}, u_0 and u_0^m, where m is to be determined. From the definition of L and using the fact that $Lu_{0x} = 0$ we have

$$L(xu_{0x}) = 2u_{0xx}, \tag{11}$$

$$L(u_0) = u_{0xx} + \sigma_A \sigma_C u_0^{p+1} + \sigma_C B u_0^{2p+1} - \frac{u_0}{p^2}, \tag{12}$$

$$\begin{aligned} L(u_0^m) = {} & m u_0^{m-1} u_{0xx} + m(m-1) u_0^{m-2} u_{0x}^2 + \sigma_A \sigma_C u_0^{p+m} \\ & + \sigma_C B u_0^{2p+m} - \frac{u_0^m}{p^2}. \end{aligned} \tag{13}$$

Since u_0 is a solution of (4), we have

$$u_{0xxx} = \left(\frac{1}{p^2} - \sigma_A \sigma_C u_0^p - \sigma_C B u_0^{2p}\right) u_{0x}$$

which on integrating gives

$$u_{0xx} = \frac{u_0}{p^2} - \frac{\sigma_A \sigma_C}{p+1} u_0^{p+1} - \frac{\sigma_C B}{2p+1} u_0^{2p+1}. \tag{14}$$

Multiplying (14) by $2u_{0x}$ and integrating again yields

$$u_{0x}^2 = \frac{u_0^2}{p^2} - \frac{2\sigma_A \sigma_C u_0^{p+2}}{(p+1)(p+2)} - \frac{\sigma_C B u_0^{2p+2}}{(2p+1)(p+1)}. \tag{15}$$

Hence we see that the right-hand sides of (11–13) can be written in terms of powers of u_0, and to minimize the number of different powers we must choose $m = p+1$. We then find that

$$L(u_0^{p+1}) = \frac{p+2}{p} u_0^{p+1} - \frac{2\sigma_A \sigma_C p}{p+2} u_0^{2p+1}.$$

Matching coefficients of the linear sum operated on by L with the right-hand side of (10) it is then straightforward to show that

$$\phi_1 = -\frac{\sigma_C p^2 \gamma_1}{2}\left(x u_{0x} + \frac{1+\alpha^2}{p} u_0 + \frac{\sigma_A (p+2)\alpha^2 B}{p(2p+1)} u_0^{p+1}\right). \tag{16}$$

To find the first-order growth rate γ_1, we must consider the equation obtained at second order in k,

$$\frac{\mathrm{d}}{\mathrm{d}x}L\phi_2 = -\sigma_C \gamma_2 \phi_0 - \sigma_C \gamma_1 \phi_1 + \phi_{0x}. \tag{17}$$

However, we do not need to find ϕ_2. As in Ref. [15], we instead first multiply (17) by u_0 and integrate over all x. The left-hand side can be seen to equal zero after integrating by parts and using the self-adjoint property of L and the fact that $L\phi_0 = 0$. The first term on the right-hand side also vanishes as the integrand is odd and we are left with

$$\sigma_C \gamma_1 \langle \phi_1 u_0 \rangle = \langle \phi_{0x} u_0 \rangle$$

where $\langle . \rangle$ is the integral over all x. Using (16) and the results $\langle u_0 u_{0xx} \rangle = -\langle u_{0x}^2 \rangle$ and $\langle x u_0 u_{0x} \rangle = -\frac{1}{2}\langle u_0^2 \rangle$ we finally obtain

$$\gamma_1^2 = \frac{4\langle u_{0x}^2 \rangle_{\mathrm{s}}}{p(2(1+\alpha^2) - p)\langle u_0^2 \rangle_{\mathrm{s}} + \dfrac{2\sigma_A p(p+2)\alpha^2 B}{2p+1}\langle u_0^{p+2} \rangle_{\mathrm{s}}} \tag{18}$$

where for later convenience we have scaled the integrals, writing $\langle . \rangle_{\mathrm{s}} \equiv \langle . \rangle / (2\beta)^{2/p}$. Notice that since $\mathrm{sgn}(u_0^{p+2}) =$

$\sigma_A\sigma_C$ and if $B \neq 0$, $\sigma_C B > 0$, the second term in the denominator of (18) is never negative. As $\alpha \geq 1$, it is only possible for the denominator to be zero overall if $p \geq 4$ in which case the solitary pulse solution is already unstable in the x-direction.

Using (5) the scaled integrals can be obtained in closed form with the help of *Mathematica* after making the substitution $w = \cosh x$. We obtain

$$\left\langle u_0^2 \right\rangle_{\rm s} = \frac{\Gamma^2(\frac{1}{p}) H_1(\frac{1}{p}, \frac{1}{p}) - 2\alpha\Gamma^2(\frac{1}{2}+\frac{1}{p}) H_3(\frac{1}{2}+\frac{1}{p}, \frac{1}{2}+\frac{1}{p})}{2\Gamma(\frac{2}{p})}, \tag{19}$$

$$\langle u_0^{p+2}\rangle_{\rm s} = \frac{\sigma_A\sigma_C\beta}{\Gamma(1+\frac{2}{p})}\Big[\Gamma^2(\tfrac{1}{2}+\tfrac{1}{p})H_1(\tfrac{1}{2}+\tfrac{1}{p},\tfrac{1}{2}+\tfrac{1}{p}) - 2\alpha\Gamma^2(1+\tfrac{1}{p})H_3(1+\tfrac{1}{p},1+\tfrac{1}{p})\Big], \tag{20}$$

$$\left\langle u_{0x}^2 \right\rangle_{\rm s} = \frac{1}{p^2\Gamma(2+\frac{2}{p})}\Big[\Gamma(1+\tfrac{1}{p})\Gamma(\tfrac{1}{p})H_1(1+\tfrac{1}{p},\tfrac{1}{p}) - 2\alpha\Gamma(\tfrac{1}{2}+\tfrac{1}{p})\Gamma(\tfrac{3}{2}+\tfrac{1}{p})H_3(\tfrac{1}{2}+\tfrac{1}{p},\tfrac{3}{2}+\tfrac{1}{p})\Big], \tag{21}$$

in which $H_n(a,b) \equiv {}_2F_1(a,b;\frac{n}{2};\alpha^2)$ where ${}_2F_1$ is the hypergeometric function. The rather complicated expression we have obtained for the first order growth rate, γ_1, simplifies in the cases $p = 1$ and $p = 1/2$ and is in agreement with the results obtained in Refs. [14] and [15] for the $(1,2)$-mZK and $(\frac{1}{2},1)$-mZK equations, respectively.

When $\alpha^2 = 1$, which corresponds to $B = 0$ and hence the generalization of the ZK equation with a single nonlinear term, (18) reduces to

$$\gamma_1^2 = \frac{4\left\langle u_{0x}^2 \right\rangle_{\rm s}}{p(4-p)\left\langle u_0^2 \right\rangle_{\rm s}} \tag{22}$$

and (19) and (21) can be simplified to give

$$\left\langle u_0^2 \right\rangle_{\rm s} = \frac{\sqrt{\pi}\Gamma(\frac{1}{2}-\frac{2}{p})}{2\Gamma(\frac{2}{p})}\left[\frac{\Gamma^2(\frac{1}{p})}{\Gamma^2(\frac{1}{2}-\frac{1}{p})} - \frac{\Gamma^2(\frac{1}{2}+\frac{1}{p})}{\Gamma^2(1-\frac{1}{p})}\right],$$

$$\left\langle u_{0x}^2 \right\rangle_{\rm s} = \left[\frac{\Gamma(\frac{1}{p})\Gamma(1+\frac{1}{p})}{\Gamma(\frac{1}{2}-\frac{1}{p})\Gamma(-\frac{1}{2}-\frac{1}{p})} - \frac{\Gamma(\frac{1}{2}+\frac{1}{p})\Gamma(\frac{3}{2}+\frac{1}{p})}{\Gamma(-\frac{1}{p})\Gamma(1-\frac{1}{p})}\right] \times \frac{\sqrt{\pi}\Gamma(-\frac{1}{2}-\frac{2}{p})}{p^2\Gamma(2+\frac{2}{p})}.$$

Putting $p = 1, 2, 1/2$ in (22) gives values of γ_1^2 of $4/15$, $4/3$, and $64/63$, respectively. After allowing for the different scalings these agree with the results in Refs. [11–13]. As $p \to 4$, the scaled integrals remain finite with the result that the growth rate diverges. This coincides with the onset of instability of the pulse with respect to perturbations in the direction of propagation.

4. Discussion

The $(p, 2p)$-mZK equation appears to be the most general form of a modified ZK equation for which an explicit expression for the first order growth rate can be obtained. In the case of the (p, q)-mZK equation (or generalizations with more nonlinearities, as have been obtained in Ref. [7]) there are no closed-form expressions for the pulse solution. Furthermore, with such generalizations the cancellations of powers of u_0 in $L(u_0^m)$ that allowed us to find ϕ_1 using the straightforward technique given here do not occur. Nevertheless, this approach should be applicable to obtaining the transverse instability growth rate of other families of equations with more than one spatial dimension whose solitary pulse solutions are known.

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