Lecture 22 Relativistic Kinematics

SCPY152 General Physics II, MUSC, Second Semester 2021-22

U. Robkob, Physics-MUSC

April 5, 2022

Relativistic kinematics

- Set space-time coordinates on M^4 as $x^\mu = (ct, \vec{x})$. It is called 4-position

$$x^2 = \tau^2 - proper\ time\ (LT - invariance)$$

- LT of 4-position, with Lorentz boost in z-direction

$$\begin{aligned} ct' &= \gamma(ct - \beta z) \\ z' &= \gamma(z - \beta ct) \qquad (1) \\ x' &= x,\ \ y' = y \qquad (2) \end{aligned}$$

- The inverse LT will be

$$\begin{aligned} ct &= \gamma(ct' + \beta z') \\ z &= \gamma(z' + \beta ct') \qquad (3) \\ x &= x',\ \ y = y' \qquad (4) \end{aligned}$$

- The 4-velocity is calculated as usual as

$$v^\mu = \frac{dx^\mu}{d\tau} \xrightarrow[\text{time dilation}]{} \gamma \frac{dx^\mu}{dt} = \gamma(c, \vec{v}), \ \vec{v} = \frac{d\vec{x}}{dt}$$

$$v^2 = \gamma^2(c^2 - \vec{v}\cdot\vec{v}) = \gamma^2 c^2 (1-\beta^2) = c^2, \vec{\beta} = \frac{\vec{v}}{c}$$

- Define 4-momentum as

$$p^\mu = mv^\mu = (\gamma mc, \gamma m\vec{v}) \equiv (E/c, \vec{p}), \ \vec{p} = \gamma m\vec{v}$$

$$E = \gamma mc^2 = K + mc^2$$

$$p^2 = \frac{E^2}{c^2} - \vec{p}\cdot\vec{p} \equiv m^2c^2 \mapsto E^2 = |\vec{p}|^2c^2 + m^2c^4$$

- Some useful formula

$$\gamma = \frac{E}{mc^2}, \ \vec{\beta} = \frac{\vec{p}}{E/c}$$

- Derivation of an expression of E, in simply 1-spatial direction,

$$dW = Fdx = \frac{dp}{dt}dx = dp\frac{dx}{dt} = vdp = vd(\gamma mv) = mc^2\beta d(\gamma\beta)$$

$$\gamma = (1-\beta^2)^{-1/2} \mapsto dW = \frac{1}{2}mc^2\gamma d\beta^2 + \frac{1}{2}\gamma^3\beta^2 d\beta^2$$

$$= \frac{1}{2}mc^2\gamma^3 d\beta^2 \equiv mc^2 d\gamma \quad (5)$$

$$\mapsto W_{0\to\beta} = \gamma mc^2 - mc^2 \equiv K_\beta - 0 \quad (6)$$

$$\mapsto E := \gamma mc^2 = K_\beta + mc^2 \quad (7)$$

- $p^\mu \in M^4$, also satisfy LT, i.e., $p^\mu \xrightarrow[LT]{} p'^\mu$ with Lorentz boost in z-direction

$$E'/c = \gamma(E/c - \beta p_z), p'_z = \gamma(p_z - \beta E/c),\ \ p'_x = p_x, p'_y = p_y$$

$$E/c = \gamma(E'/c + \beta p'_z), p_z = \gamma(p'_z + \beta E'/c),\ \ p_x = p'_x, p_y = p'_y$$

Massive particle decay

- Decay of massive particle $X \rightarrow a + b$, in the rest frame of X, we apply conservation on momentum and energy conditions as

$$p_a = p_b = p \text{ in } \hat{\mathrm{n}} - \mathrm{direction} \tag{8}$$

$$\begin{aligned} E_X = E_a + E_b \mapsto M_X c^2 &= \sqrt{p_a^2 c^2 + m_a^2 c^4} + \sqrt{p_b^2 c^2 + m_b^2 c^4} \\ &= \sqrt{p^2 c^2 + m_a^2 c^4} + \sqrt{p^2 c^2 + m_b^2 c^4} \end{aligned} \tag{9}$$

$$\underbrace{\left[(M_x c^2)^2 - (m_a c^2)^2 - (m_b c^2)^2\right]}_{A^2} - 2p^2 c^2$$

$$= 2\sqrt{(p^2 c^2 + m_a^2 c^4)(p^2 c^2 + m_b^2 c^4)} \tag{10}$$

$$A^4 - 4A^2 p^2 c^2 + 4p^4 c^4 = 4(p^2 c^2 + m_a^2 c^4)(p^2 c^2 + m_b^2 c^4)$$

$$\mapsto A^4 - 4m_a^2 m_b^2 c^2 = 4[(m_a^2 + m_b^2)c^4 + A^2]p^2 c^2 = 4(M_X^2 c^4)p^2 c^2 \tag{11}$$

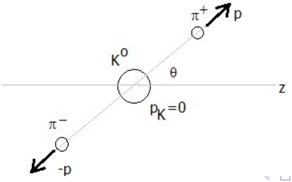
$$\mapsto pc = \frac{1}{2M_x c^2}\sqrt{A^4 - 4m_a^2 m_b^2 c^4} \tag{12}$$

- Cont.

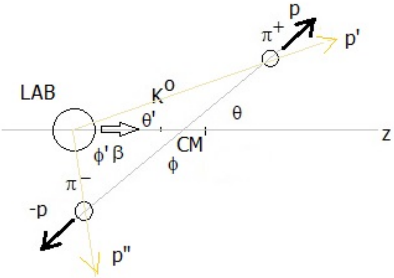
$$pc = \frac{1}{2M_x c^2}\sqrt{((M_X c^2)^2 - (m_a c^2 + m_b c^2)^2)}$$
$$\times\sqrt{((M_x c^2)^2 - (m_a c^2 - m_b c^2)^2)} \quad (13)$$
$$m_a = m_b = m \mapsto pc = \frac{1}{2}\sqrt{(M_x c^2)^2 - 2m^2c^4} \quad (14)$$

- For example $K^0 \to \pi^+ + \pi^-$, $M(K^0) = 498 MeV/c^2$ and $m(\pi^\pm) = 140 MeV/c^2$. In Kaon rest frame (CM-frame)

$$p_K = 0,\ \ p_\pi = \frac{1}{2}\sqrt{(498)^2 - 2(140)^2} = 228 MeV/c$$
$$E_\pi = \sqrt{(228)^2 + (140)^2} = 267 MeV \mapsto \frac{E_\pi}{p_\pi c} = \frac{267}{228} = 1.17$$



- Kaon decay (cont.). In the LAB frame, let Kaon has energy $E'_K = 1.0 GeV$ and moves in z-direction



$$E'_K = 1000 MeV \mapsto p'_K c = \sqrt{(1000)^2 - (498)^2} = 867 MeV \quad (15)$$

$$\mapsto \beta = \frac{p'_K c}{E'_K} = 0.867, \gamma = \frac{E'_K}{M_K c^2} = \frac{1000}{498} \simeq 2 \quad (16)$$

- Kaon decay (cont.), for the π^+,

$$\begin{aligned}
\mathrm{CM-frame} \quad & p^\mu = (E_\pi/c, p\sin\theta, 0, p\cos\theta) && (17)\\
\mathrm{LAB-frame} \quad & p'^\mu = (E'_\pi/c, p'\sin\theta', 0, p'\cos\theta') && (18)\\
LT \quad & E'_\pi/c = \gamma(E_\pi/c + \beta p\cos\theta) && (19)\\
& p'\cos\theta' = \gamma(p\cos\theta + \beta E_\pi/c) && (20)\\
& p'\sin\theta' = p\sin\theta && (21)
\end{aligned}$$

$$\mathrm{Fix}\ \theta \mapsto \tan\theta' = \frac{\sin\theta}{\gamma\cos\theta + \beta(E_\pi/pc)} \quad (22)$$

Let $\theta = 90^0 = \phi$

$$\tan\theta' = \frac{1}{(0.867)(1.17)} = 0.986 \mapsto \theta' = 44.6^0 = \phi' \quad (23)$$

$$p' = \frac{228 MeV/c}{\sin 44.6^0} = 325 MeV/c = p'' \quad (24)$$

$$E'_\pi = \sqrt{(p'c)^2 + (m_\pi c^2)^2} = 354 MeV = E''_\pi \quad (25)$$

- Kaon decay in the Lab frame