Lecture 3 Nuclear Models I

SCPY327 ANP, Physics-MUSC, 2021/2

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Today topics

- Schrodinger's equation of central potential
- Shell models
  - Rigid sphere potential
  - Isotropic harmonic potential
  - Spin-orbit coupling

Schrodinger's equation of central potential

- 3D Schrodinger equation in spherical system, i.e., $\vec{r} = (r, \theta, \phi)$

$$\nabla^2 \varphi_E(\vec{r}) + \frac{2m}{\hbar^2}(E - U(r))\varphi_E(\vec{r}) = 0 \quad (1)$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\} \quad (2)$$

$$\varphi_E(\vec{r}) = \varphi_E(r, \theta, \phi) = R_{E,\alpha}(r) Y_{\alpha\beta}(\theta, \phi) \quad (3)$$

Insertion into (1), we get

$$\frac{1}{R_{E,\alpha}}\frac{d}{dr}\left(r^2 \frac{dR_{E,\alpha}}{dr}\right) + \frac{2mr^2}{\hbar^2}(E - U(r)) = \alpha^2$$
$$= -\frac{1}{Y_{\alpha,\beta}}\left(\frac{1}{\sin\theta}\frac{\partial Y_{\alpha,\beta}}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{\alpha,\beta}}{\partial\phi^2}\right) \quad (4)$$

- Solution for $Y_{\alpha\beta}(\theta,\phi)$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{\alpha,\beta}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{\alpha,\beta}}{\partial\phi^2}+\alpha^2 Y_{\alpha,\beta}=0 \quad (5)$$

$$Y_{\alpha,\beta}(\theta,\phi)=P_{\alpha,\beta}(\theta)\Phi_\beta(\phi) \quad (6)$$

$$(5)\mapsto \frac{1}{P_{\alpha,\beta}}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dP_{\alpha,\beta}}{d\theta}\right)\right]+\alpha^2\sin^2\theta$$

$$=-\frac{1}{\Phi_\beta}\frac{d^2\Phi_\beta}{d\phi^2}=m^2 \quad (7)$$

$$\Phi_m(\phi)=e^{im\phi},\ \ \beta=m=0,\pm1,\pm2,\ldots \quad (8)$$

$$(7)\mapsto \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_{\alpha,m}}{d\theta}\right)+(\alpha^2-\frac{m^2}{\sin^2\theta})P_{\alpha,m}=0 \quad (9)$$

Let $y=\cos\theta\mapsto dy=-\sin\theta d\theta$ and $P_{\alpha,m}=P_{\alpha,m}(y)$

$$\frac{d}{dy}\left((1-y^2)\frac{dP_{\alpha,m}}{dy}\right)+\left(\alpha^2-\frac{m^2}{1-y^2}\right)P_{\alpha,m}=0$$

$$(1-y^2)\frac{d^2P_{\alpha,m}}{dy^2}-2y\frac{dP_{\alpha,m}}{dy}+\left(\alpha^2-\frac{m^2}{1-y^2}\right)P_{\alpha,m}=0 \quad (10)$$

- Equation (10) is known from Wolfram Alpha as *associated Legendre equation*, its solution is known in the name of *associated Legendre polynomials* $P_{l,m}(y)$, with conditions

$$\alpha = l = 0, 1, 2, 3, ...$$

$$m = 0, \pm 1, \pm 2, ..., \pm l$$

Back to (6), we get angular solution in the form

$$Y_{l,m}(\theta, \phi) = N_{lm} P_{lm}(\cos\theta) e^{im\phi} \tag{11}$$

$$N_{lm} = \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}}, \ Y_{l,-m} = (-1)^m Y^*_{l,m} \tag{12}$$

$$\int_0^{4\pi} d\Omega Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'} \tag{13}$$

It is known as *spherical harmonic.*, normally it is used to be a basis set for all angular functions.

- Back to (4), radial equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_{El}}{dr}\right)+\left(\frac{2mE}{\hbar^2}-\frac{2mU(r)}{\hbar^2}-\frac{l(l+1}{r^2}\right)R_{El}=0$$

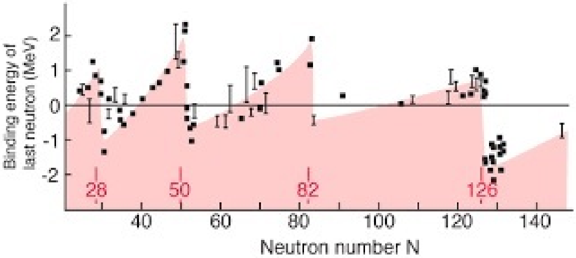
$$\frac{d^2R_{El}}{dr^2}+2r\frac{dR_{El}}{dr}+\left(\frac{2mE}{\hbar^2}-\frac{2mU(r)}{\hbar^2}-\frac{l(l+1}{r^2}\right)R_{El}=0(14)$$

Equation (14) will be solved for different central potential term $U(r)$ of nuclear shell models.

Nuclear Shell Models

- Nuclear shell model follows the same philosophy of shell structure of atoms, this comes from the appearance of *magic numbers* of the proton or neutron numbers at which the nuclei have exceptional stability.

$$Z, N = 2, 8, 20, 28, 50, 82 \text{ and } N = 126$$

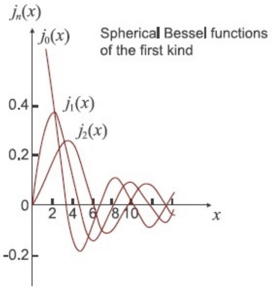


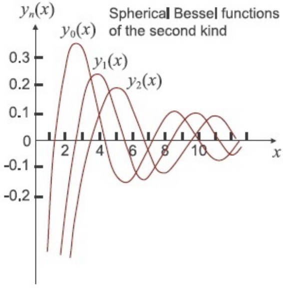
- There is no direct central potential derived from nuclear strong force, but some central potentials are used to determined the possible shell structure of the nucleus

Shell Model: Rigid Sphere Potential

- Rigid sphere potential, with the radius a,
$$U(r) = \begin{cases} 0, & 0 \leq r \leq a \\ \infty, & r > a \end{cases}$$
Solution of (14), inside the sphere will be
$$R''_{El} + \frac{2}{r}R'_{El} + \left(\frac{2mE}{\hbar^2} - \frac{l(l+1)}{r^2}\right)R_{El} = 0 \tag{15}$$
$$k^2 = \frac{2mE}{\hbar^2} \mapsto \rho = kr,\ R_{El} = R_{El}(rho)$$
$$(15) \mapsto \rho^2 R''_{El} + 2\rho R'_{El} + (\rho^2 - l(l+1))R_{El} = 0 \tag{16}$$
- Equation (16) is known from Wolfram Alpha as *spherical Bessel equation*, its solutions are *spherical Bessel function of the first kind* $j_l(\rho)$ and *the second kind* $n_l(\rho)$
$$R_{El}(kr) = C_{1l}j_l(kr) + C_{2l}n_l(kr) \tag{17}$$
But $n_l(kr)$ is not finite at origin $r = 0$, so the we set $C_{2l} = 0$, and we have only $R_{El}(r) = C_{1l}j_l(kr)$ as our

- Spherical Bessel functions



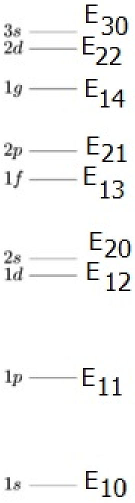


- Boundary condition at $r = a \mapsto j_l(ka) = 0$, then k will be determined from the *zero* of the spherical Bessel function of the first kind, i.e., $ka = Z_{nl}$ (*https://keisan.casio.com/exec/system/1180573465*)

n	$j_0(x)$	$j_1(x)$	$j_2(x)$	$j_3(x)$	$j_4(x)$
1	3.14159	4.49341	5.76346	6.98793	8.18256
2	6.28319	7.72525	9.09501	10.4171	11.7049
3	9.42478	10.9041	12.3229	13.6980	15.0397
4	12.5664	14.0662	15.5146	16.9236	18.3013
5	15.7080	17.2208	18.6890	20.1218	21.5254

- Nuclear energy levels are

$$ka = Z_{nl} \mapsto E = E_{nl} = \frac{\hbar^2 Z_{nl}^2}{2ma^2} \tag{18}$$



Shell Model: Isotropic Harmonic Potential

- Isotropic harmonic potential

$$U(r) = \frac{1}{2} m\omega r^2 \tag{19}$$

Solution of (14) will be

$$R''_{El} + \frac{2}{r} R'_{el} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2 r^2}{\hbar^2} - \frac{l(l+1)}{r^2} \right) R_{El} = 0 \tag{20}$$

$$\alpha^2 = \frac{m\omega}{\hbar}, \;\; \epsilon = \frac{2E}{\hbar\omega}, \rho = \alpha r, \;\; R_{El} = R_{El}(\rho)$$

$$(20) \mapsto R''_{El} + \frac{2}{\rho} R'_{El} + \left(\epsilon - \rho^2 - \frac{l(l+1)}{\rho^2} \right) R_{El} = 0 \tag{21}$$

At $\rho \to \infty$,

$$(22) \to R''_{El} - \rho^2 R_{El} \sim 0 \mapsto R_{El}(\rho) \sim e^{-\rho^2/2}$$

At $\rho \to 0$,

$$(22) \to R''_{El} + \frac{2}{\rho} R'_{El} - \frac{l(l+1)}{\rho^2} \sim 0 \mapsto R_{El}(\rho) \sim \rho^l$$

- Let us define

$$R_{El}(\rho) = \rho^l \sum_{k=0}^{\infty} a_k \rho^k e^{-\rho^2/2} \tag{22}$$

$$\begin{aligned} R'_{El} &= \sum_k a_k \left[(l+k)\rho^{l+k-1} - \rho^{l+k+1}\right] e^{-\rho^2/2} \\ R''_{El} &= \sum_k a_k \Big[(l+k)(l+k-1)\rho^{l+k-2} \\ &\quad -(2l+2k+1)\rho^{l+k} + \rho^{l+k+2}\Big] e^{-\rho^2/2} \end{aligned} \tag{23}$$

Insertion into (21), we get

$$\begin{aligned} \sum_{k=0}^{\infty} a_k \Big[&(l+k)(l+k-1)\rho^{l+k-2} - (2l+2k+1)\rho^{l+k} \\ &+\rho^{l+k+2} + 2(l+k)\rho^{l+k-2} - 2\rho^{l+k} \\ &+\epsilon\rho^{l+k} - \rho^{l+k+2} - l(l+1)\rho^{l+k-2}\Big] = 0 \end{aligned} \tag{24}$$

- Continue from (24)

$$\sum_{k=0}^{\infty} a_k \Big[\{(l+k)(l+k-1) + (2(k+l) - l(l+1)\} \rho^{l+k-2}$$
$$+ \{\epsilon - (2l+2k+1) - 2\} \rho^{l+k} \Big] = 0$$

$$\mapsto \sum_{k=0}^{\infty} a_k \left[k(2l+k+1)\rho^{l+k-2} + \{\epsilon - (2l+2k+3)\} \rho^{l+k} \right] = 0 \quad (25)$$

There are two series of (25), i.e., even $k = 0, 2, 4, ...$ and odd $k = 1, 3, 5, ...$, corresponding to two solutions of the equation.

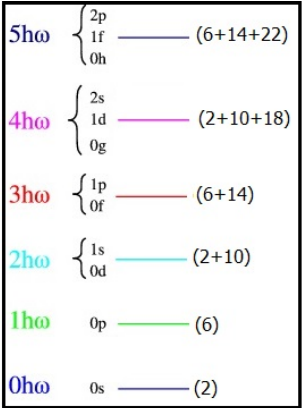
To terminate the series, we will have a condition

$$\epsilon = \frac{2E}{\hbar\omega} = 2l + 2k + 3 \mapsto E_{kl} = (k + l + 3/2)\,\hbar\omega$$
$$k = n_r = 0, 1, 2, ... \mapsto E_{n_r l} = (n_r + l + 3/2)\hbar\omega \qquad (26)$$
$$(25) \mapsto a_{k+2} = -\frac{E_{n_r l} - (2l+2k+3)}{(k+2)(2l+k+3)} a_k$$
$$= \frac{2(k-n_r)}{\cdots} a_k \qquad (27)$$

- To be convenience, let us shift $n_r \to 2n_r$ and $k \to 2k$, then we have from (26,27)

$$E_{n_r l} = (2n_r + l + 3/2)\hbar\omega \tag{28}$$

$$a_{k+1} = \frac{k - n_r}{(k+1)(l + k + 3/2)} a_k \tag{29}$$



Shell Model: Spin-Orbit Coupling

- LS-coupling potential, derived in atomic physics, is

$$U_{LS} = -\frac{1}{2m^2c^2}\frac{1}{r}\frac{dV_c(r)}{dr}\left(\vec{L}\cdot\vec{S}\right) \xrightarrow{NP} -\alpha(r)\left(\vec{L}\cdot\vec{S}\right)$$

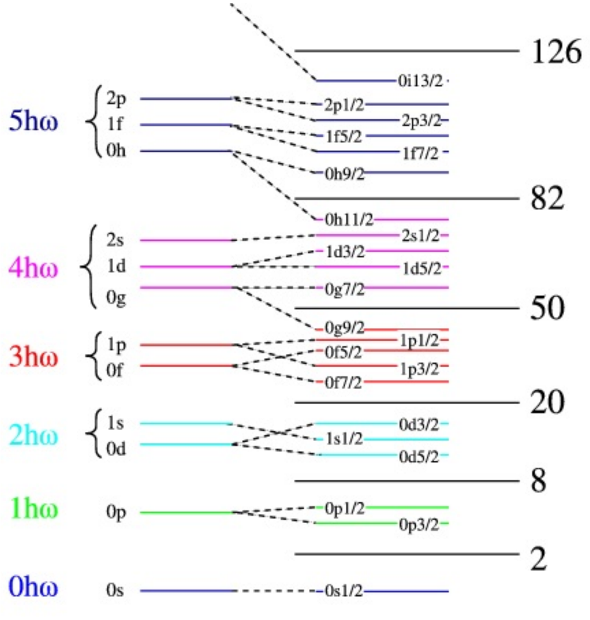
- Let us determine

$$\begin{aligned} \vec{J} &= \vec{L}+\vec{S} \mapsto J^2 = j(j+1)\hbar^2 && (30) \\ \vec{L}\cdot\vec{S} &= \frac{1}{2}\left(J^2 - L^2 - S^2\right) \\ &= \frac{\hbar^2}{2}\left(j(j+1) - l(l+1) - s(s+1)\right) && (31) \end{aligned}$$

Since $s = 1/2,\ j = l+1/2, l-1/2$, this leads to

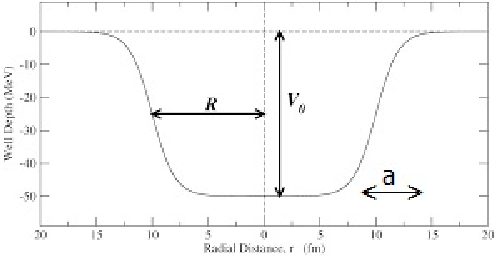
$$\vec{L}\cdot\vec{S} = \frac{\hbar^2}{2}\begin{cases} l, & j = l+1/2 \\ -l-1, & j = l-1/2 \end{cases} \tag{32}$$

- LS-splitting in nuclear shell model (isotropic HO)



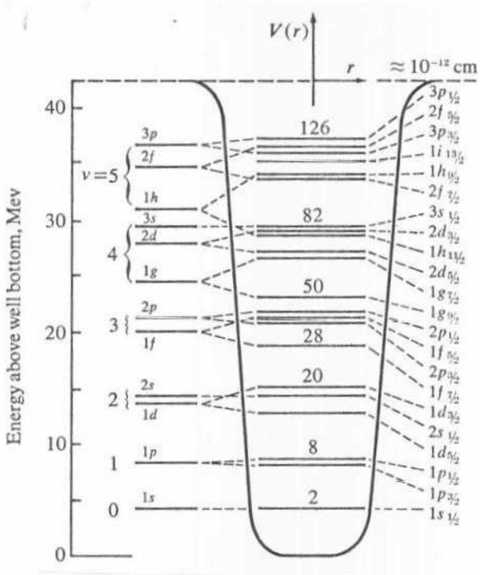
Woods-Saxon potential

- Woods-Saxon potential
$$V_{WS}(r) = -\frac{V_0}{1 + e^{\frac{r-R}{a}}}$$
with radius $R = r_0 A^{1/3}$, the potential depth $V_0 \sim 50MeV$, and the skin thickness $a \sim 0.5fm$.



- Non-analytic solution of radial function of Schrodinger's equation appear in Flugge (1994), *Practical Quantum Mechanics (Springer Verlag)*, Problem 64.

- Analytic solution of radial function of SE is derived with *Nikiforov-Uvarov (NU) method*
- Energy levels + LS coupling (cannot find the reference yet)



- Predictions of shell model
    - magic numbers
    - explain spin and ground state parity of most nuclei
    - explain nuclear dipole and quadrupole moments of most nuclei
- Failures of the shell model
    - some nuclei cannot get predictions from shell model