Lecture 6 Beta Emission

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Today topics

- Fermi golden rule
- Beta emission

Fermi golden rule

- Basic Schrodinger equation

$$\hat{H}_0|n\rangle = E_n|n\rangle$$

- With time-dependent interaction $H(t) = H_0 + V(t)$, the Schrodinger equation in interaction picture is

$$i\hbar\partial_t|\psi(t)\rangle_S = \hat{H}(t)|\psi(t)\rangle_S,\ |\psi(t)\rangle_S = \sum_n c_n(0)e^{-iE_nt/\hbar}|n\rangle \quad (1)$$

$$|\psi(t)\rangle_I = e^{iH_0t/\hbar}|\psi(t)\rangle_S,\ V_I(t) = e^{iH_0t/\hbar}V(t)e^{-iH_0t/\hbar} \quad (2)$$

$$\mapsto i\hbar\partial_t|\psi(t)\rangle_I = V_I(t)|\psi(t)\rangle_I \quad (3)$$

$$|\psi(t)\rangle_I = U_I(t,0)|\psi(0)\rangle_I \mapsto U_I(t,0) = e^{-i\int_0^t V_I(t')dt'/\hbar} \quad (4)$$

$$|\psi(t)\rangle_I = \sum_n |n\rangle \underbrace{\langle n|U_I(t,0)|\psi(0)\rangle_I}_{c_n(t)} \quad (5)$$

- Perturbation theory

$$
\begin{aligned}
\mapsto c_n(t) &= c_n^{(0)}(0) + c_n^{(1)}(t) + c_n^{(2)}(t) + \ldots & (6)\\
c_n^{(0)}(0) &= \langle n|\psi(0)\rangle_I \equiv \delta_{nm},\ |\psi(0)\rangle_I = |m\rangle & (7)\\
c_n^{(1)}(t) &= -\frac{i}{\hbar}\int_0^t \langle n|V_I(t')|m\rangle dt' \\
&= -\frac{i}{\hbar}\int_0^t V_{nm}(t') e^{i\omega_{nm}t'} dt' & (8)\\
P_n(t) &= |c_n(t)|^2 \simeq |c_n^{(1)}(t)|^2 & (9)
\end{aligned}
$$

- Harmonic perturbation

$$
\begin{aligned}
V(t) = Ve^{-i\omega t} \mapsto c_f^{(1)}(t) &= -\frac{i}{\hbar}\int_0^t V_{fi} e^{i(\omega_{fi}t-\omega)t'} dt' \\
&= -\frac{i}{\hbar} V_{fi} \frac{e^{i(\omega_{fi}-\omega)t} - 1}{i(\omega_{fi}-\omega)} & (10)
\end{aligned}
$$

$$
P_{i\to f} = \frac{1}{\hbar^2}|V_{fi}|^2 \left(\frac{\sin((\omega_{fi}-\omega)t/2)}{(\omega_{fi}-\omega)/2}\right)^2 \simeq \frac{2\pi t}{\hbar^2}|V_{fi}|^2 \delta(\omega_{fi}-\omega)
$$

- The first Fermi golden rule of transition rate

$$W_{i\to f} = \frac{P_{i\to f}}{t} = \frac{2\pi}{\hbar^2}|v_{fi}|^2\delta(\omega_{fi} - \omega) \tag{11}$$

- Transition to a group of final states with density of state $\rho(E_f)$

$$\begin{aligned} W_{i\to f} &= \frac{2\pi}{\hbar^2}|V_{fi}|^2 \int \rho(E_F)\delta(\omega_{fi} - \omega)dE_F & (12) \\ &\simeq \frac{2\pi}{\hbar}|Vfi|^2\rho(E_F) & (13) \end{aligned}$$

The second Fermi golden rule.

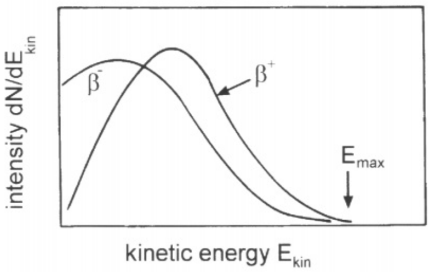
Beta emission

- $\beta^{\pm} = \pm e$
- Decay equations

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + \beta^{-} + \bar{\nu}_e + Q_{\beta^-} $$

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z-1}Y + \beta^{+} + \nu_e + Q_{\beta^+} $$

  These come from fundamental decays $n^0 \rightarrow p^+1 + \beta^- + \bar{\nu}_e$ and $p^+ \rightarrow n^0 + \beta^+ + \nu_e$
- Beta spectrum and the existence of *neutrino/anti-neutrino* (W.Pauli, 1930)



- Fermi theory of beta emission
- The number of state of fermion having momentum between $p \to p + dp$ inside a volume V is

$$dN = \frac{V}{2\pi^2\hbar^3} p^2 dp \mapsto \frac{dN}{dE} = \frac{V}{2\pi^2\hbar^3} p^2 \frac{dp}{dE}$$

For the neutrino

$$p_\nu = E_\nu / c \mapsto dp_\nu = \frac{dE_\nu}{c} \mapsto \frac{dN_\nu}{dE_\nu} = \frac{V}{2\pi^2\hbar^3 c^3} E_\nu^2 \tag{14}$$

For the beta

$$p_e c = \sqrt{E_e^2 - (m_e c^2)^2} \mapsto dp_e c = \frac{E_e dE_e}{\sqrt{E_e^2 - (m_e c^2)^2}} \tag{15}$$

$$\mapsto \frac{dN_e}{dE_e} = \frac{V}{2\pi^2\hbar^3 c^3} \sqrt{E_e^2 - (m_e c^2)^2} E_e \tag{16}$$

$$\mapsto \frac{dN_e}{dK_e} = \frac{V}{2\pi^2\hbar^3 c^3} \sqrt{K_e(K_e + 2m_e c^2)}(K_e + m_e c^2) \tag{17}$$

with the fact that $E_e = K_e + m_e c^2$

- From Fermi golden rule, with neutrino momentum q

$$\frac{dW_{e,fi}}{dp_e} = \frac{2\pi}{\hbar}|V_{fi}|^2 \frac{dN_e}{dp_e}\frac{dN_\nu}{dE_{fi}} \quad (18)$$

$$E_{fi} = K_e + cq \equiv Q \mapsto dE_{fi} = cdq \ (fixed\ \ K_e) \quad (19)$$

$$\mapsto \frac{dW_{e,fi}}{dp_e} = \frac{2\pi}{\hbar}|V_{fi}|^2 \frac{V}{2\pi^2\hbar^3} p_e^2 \frac{V}{2\pi^2\hbar^3} q^2 \delta(E_{fi} - (K_e + K_\nu)) \quad (20)$$

  The delta function is inserted to account the energy conservation.

- Let $M_{fi} = VV_{fi}$ and $qc = (Q - K_e)$, after integrate (20) over all neutrino momenta using delta function, we will have

$$\frac{dW_{e,fi}}{dp_e} = \frac{|M_{fi}|^2}{2\pi^3\hbar^7 c^3} p_e^2 (Q - K_e)^2$$

  This expression was first derived by Fermi, and display the beta spectrum.

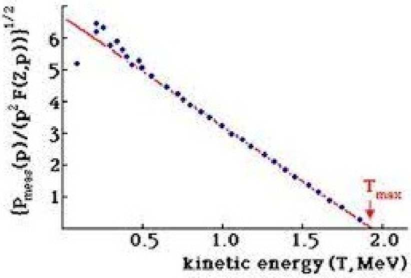
- More details analysis of M_{fi} from weak theory will lead Fermi expression to

$$\frac{dW_{e,fi}}{dp_e} \propto |M_{fi}|^2 F(Z, p_e) p_e^2 (Q - K_e)^2$$

$$N_e \simeq F(Z, p_e) p_e^2 (Q - K_e)^2$$

where $F(Z, p_e)$ is known as *Fermi function*.

- Kurie plot of beta spectrum



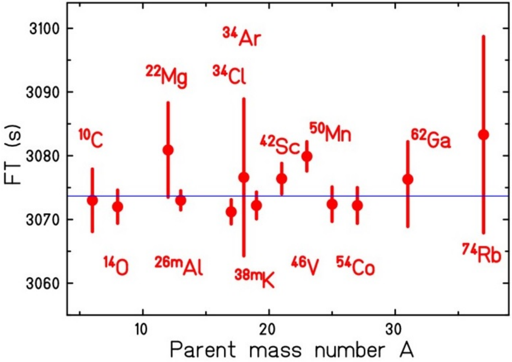
- Decay constant

$$\lambda = \int dW_{e,fi} = \frac{1}{2\pi^3\hbar^7 c^3}\int_0^{p_{e,max}} |M_{fi}|^2 F(Z,Q) p_e^2 (Q-K_e)^2 dp_e$$

$$\mapsto \lambda \simeq \frac{|M_{fi}|^2}{2\pi^3\hbar^7 c^3} f(Z,Q) \quad \text{"Fermi integral"} \quad (21)$$

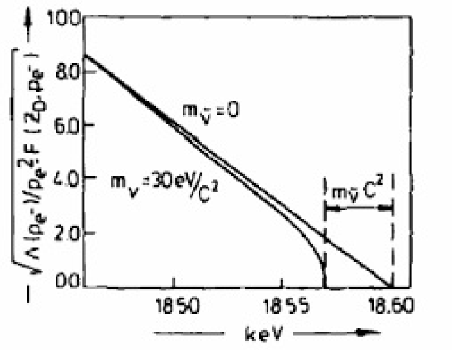
$$\mapsto ft_{1/2} = 0.693\frac{2\pi^3\hbar^7 c^3}{|M_{fi}|^2} \quad \text{"comparative half life"} \quad (22)$$

- ft-value



Neutrino mass

- The existence of neutrino mass appear in Kurie spectrum



$$N_e \propto F(Z, p_e) p_e^2 (Q - K_e) \sqrt{(Q - K_e)^2 - m_\nu^2 c^4}$$

Parity violation

- Tsung-Dao Lee and Chen-Ning Yang, the theoretical physicists who originated the idea of parity nonconservation and proposed the experiment, received the 1957 Nobel Prize in physics for this result. Chien-Shiung Wu's role in the discovery was mentioned in the Nobel prize acceptance speech, but was not honored until 1978, when she was awarded the first Wolf Prize.
- Parity violation