1 Nuclear Phenomenology

ICPY473 Nuclear Physics
MUIC, Third Trimester 2020-21

U. Robkob, Physics-MUSC

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Today Topics

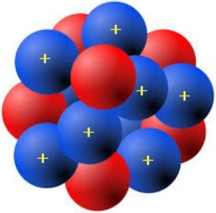
- Brief nuclear histories
- Nuclear symbol
- Nuclear shape and size
- Nuclear stability
- Nuclear decays
- Radioactivity
- Radioactive dating

Brief Nuclear Histories

- 1896: discovery of radioactivity by Becquerel
- 1898: separation of Radium by Maria and Pierre Curie; discovery of α, β, γ rays
- 1911: discovery of atomic nucleus as a central part of an atom by Rutherford
- 1919: Rutherford carries out first nuclear reaction $He + N \to p + O$
- 1928: quantum tunneling theory of alpha decay by Gamow
- 1930: prediction of the existence of neutrino by Pauli and prediction of existence of antimatter by Dirac
- 1932: discovery of the neutron (the missing part of nuclear matter) by Chadwick and discovery of positrons (anti-electron) by Anderson
- 1934: quantum theory of beta decay by Fermi, know as Fermi golden's rule

Nuclear Symbol

- Atomic nucleus consist of number of protons and neutrons, bind together by nuclear strong force



- For a nucleus of atom X with Z protons and N neutrons, its nuclear symbol is

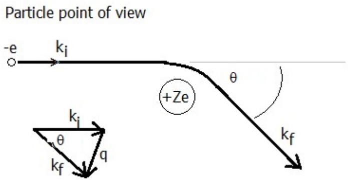
$$ {}^{A}_{Z}X_N, \quad {}^{A}_{Z}X, \quad {}^{A}X \tag{1} $$

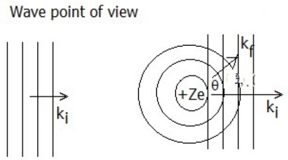
where $A = Z + N$ is the atomic mass number.

- Groups of atomic nuclei with some numbers of Z, N and A are classified to be
  - **isotope**: for the nuclei with the same Z but differ in N, A
  - **isotone**: for the nuclei with the same N but differ in Z, A
  - **isobar**: for the nuclei with the same A but differ in Z, N

Nuclear Shape and Size

- We study nuclear shape and size by using high energy electron scattering experiment
- Let us determine quantum scattering theory of high energy electron from atomic nucleus with Z protons, the interaction is Coulomb attraction, it is elastic scattering with momentum transfer $\vec{q} = \vec{k}_f - \vec{k}_i$, with $k_f = k_i = k$, so that $q = 2k\cos(\theta/2)$





- One can write

$$\phi_i(\vec{r}) \sim e^{ik_i z} \to \vec{J_i} = \frac{\hbar k_i}{m}\hat{z} \qquad (2)$$

$$\phi_f(\vec{r}) \sim f(\theta,\phi)\frac{e^{i\vec{k}_f\cdot\vec{r}}}{|\vec{r}|} + e^{ik_i z} \to \vec{J_f} = \frac{\hbar k_f}{m}|f(\theta,\phi)|^2\hat{r} \qquad (3)$$

- The differential cross section is defined to be

$$\begin{aligned} d\sigma &= \frac{\text{scattering rate in to solid angle } d\Omega}{\text{incident particle flux density}} \\ &= \frac{J_f d\Omega}{J_i} = |f(\theta,\phi)|^2 d\Omega, \quad \text{since} \quad k_f = k_i \qquad (4) \\ \frac{d\sigma}{d\Omega} &= |f(\theta,\phi)|^2 \qquad (5) \end{aligned}$$

It is measured in unit of *barn: b*, where $1.0b = 10^{-24}cm^2$

- The scattering amplitude $f(\theta,\phi)$ can be derived from the first Born approximation

$$
\begin{aligned}
f(\theta,\phi) &= -\frac{m}{2\pi\hbar^2}\langle\phi_f|V|\phi_i\rangle \\
&= -\frac{m}{2\pi\hbar^2}\int \phi_{k_f}(\vec{r})V(\vec{r})\phi_{k_i}(\vec{r})d\vec{r} \\
&= -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}}V(\vec{r})d\vec{r}
\end{aligned} \tag{6}
$$

Since

$$V(\vec{r}) = -Z\tilde{e}^2\int\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d\vec{r}', \quad \tilde{e}^2 = \frac{e^2}{4\pi\epsilon_0} \tag{7}$$

With $\vec{R} = \vec{r} - \vec{r}'$, we will have

$$f(\theta,\phi) = \frac{mZ\tilde{e}^2}{2\pi\hbar^2}\int\frac{e^{-i\vec{q}\cdot\vec{R}}}{|\vec{R}|}d\vec{R}\left(\int e^{-i\vec{q}\cdot\vec{r}'}\rho(\vec{r}')d\vec{r}'\right) \tag{8}$$

- The *form factor* is defined as

$$F(q) = \int e^{-i\vec{q}\cdot\vec{r}'}\rho(\vec{r}')d\vec{r}' \tag{9}$$

And the other integral in (8) can be calculated separately as

$$\begin{aligned}
&\int \frac{e^{-i\vec{q}\cdot\vec{R}}}{|\vec{R}|}d\vec{R} \to \int \frac{e^{-i\vec{q}\cdot\vec{R}-\alpha R}}{|\vec{R}|}d\vec{R} \\
&= 2\pi \int_0^\infty dRRe^{-\alpha R}\int_{-1}^{+1} e^{-iqR\cos\theta}d\cos\theta \\
&= \frac{2\pi}{-iq}\int_0^\infty dRe^{-\alpha R}\left(e^{-iqR} - e^{+iqR}\right) \\
&= \frac{2\pi}{-iq}\left(\frac{1}{\alpha+iq} - \frac{1}{\alpha-iq}\right) = \frac{4\pi}{\alpha^2+q^2} \\
&\xrightarrow[\alpha=0]{} \frac{4\pi}{q^2}
\end{aligned} \tag{10}$$

- Now eq.(8) becomes

$$
\begin{aligned}
f(\theta,\phi) &= \frac{2mZ\tilde{e}^2}{\hbar^2}\frac{F(q)}{q^2} = \frac{mZ\tilde{e}^2}{2\hbar^2 k^2\cos^2(\theta/2)}F(q) \\
&= \frac{Z\tilde{e}^2}{4E\cos^2(\theta/2)}F(q), \quad \text{with} \quad E = \frac{\hbar^2 k^2}{2m}
\end{aligned}
\tag{11}
$$

- The differential cross section becomes

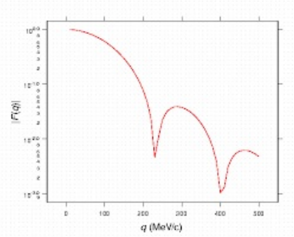
$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{Z\tilde{e}^2}{4E}\right)^2 \frac{1}{\cos^4(\theta/2)}|F(q)|^2 \\
&= \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} |F(q)|^2
\end{aligned}
\tag{12}
$$

- Let us evaluate the form factor $F(q)$ from a simple uniform sphere particle distribution

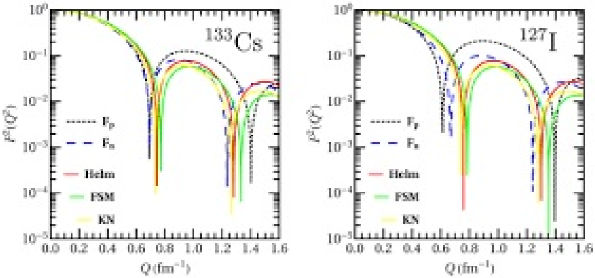
$$
\begin{aligned}
\rho(r) &= \begin{cases} \frac{3}{4\pi a^3}, & r \leq a \\ 0, & r > a \end{cases} \qquad (13) \\
F(q) &= \int e^{-i\vec{q}\cdot\vec{r}}\rho(r)d\vec{r} = 4\pi\int_0^\infty \frac{\sin(qr)}{qr}\rho(r)r^2dr \\
&= \frac{3}{qa^3}\int_0^a \sin(qr)rdr \\
&= \frac{3}{q^3a^3}\left\{-\left[x\cos(x)\right]_0^{qa} + \int_0^{qa}\cos(x)dx\right\} \\
&= -\frac{3}{q^3a^3}\left\{qa\cos(qa) - \sin(qa)\right\} \qquad (14) \\
|F(q)|^2 &= \frac{9}{q^6a^6}\left(qa\cos(qa) - \sin(qa)\right)^2 \qquad (15)
\end{aligned}
$$

with $q = 2k\cos(\theta/2)$.

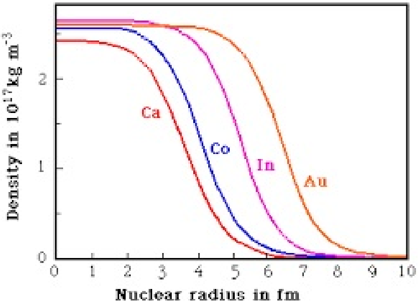
- An appearance of $|F(q)|$ is looked like a diffraction pattern as



- From experiment



- This results to the fact that atomic nucleus has uniform positive charge (or mass) distribution



- The empirical formula of the nuclear radius is then can be derived to be

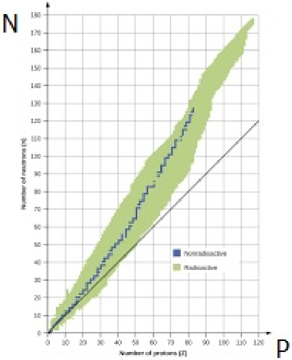
$$r = r_0 A^{1/3}, \quad r_0 = 1.2 fm \tag{16}$$

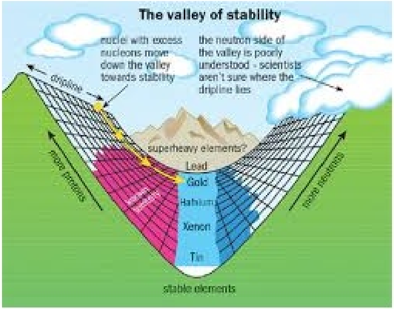
- For examples see figure below, where R is from (16) and R_p with a correction of finite nuclear size effect to Coulomb potential

Nuclide: ${}^{A}X_{Z}$	R (fm)	R_p (fm)	R_p/R
${}^{1}H_{1}$	1.2700	2.1997	1.732
${}^{7}Li_{3}$	2.4294	4.2078	1.732
${}^{23}Na_{11}$	3.6117	6.2556	1.732
${}^{39}K_{19}$	4.3068	7.4596	1.732
${}^{63}Cu_{29}$	5.0534	8.7527	1.732
${}^{85}Rb_{37}$	5.5839	9.6716	1.732
${}^{107}Ag_{47}$	6.0293	10.4431	1.732
${}^{133}Cs_{55}$	6.4827	11.2284	1.732

Nuclear Stability

- The observed nuclei in nature appear as in the following figure. Their distribution appear in the form of the *Valley of Stability*





- We can determine the nuclear stability from its binding energy calculated from a mass defect formula. For the X nucleus with Z protons and N neutrons, it is

$$\Delta M(X) = ZM(^1H) + Nm_n - M(x) \quad (17)$$
$$E_b = \Delta Mc^2 \quad (18)$$

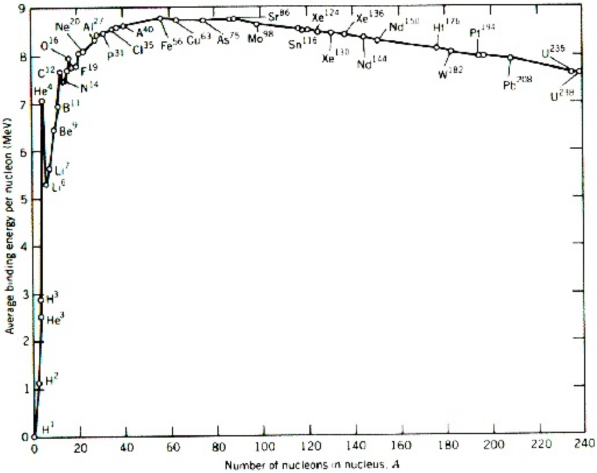
All masses are measured in atomic mass unit u, where

$$1.0u = 1.66 \times 10^{-27} kg \quad (19)$$
$$\rightarrow uc^2 = 1.49 \times 10^{-10} J = 931.5 MeV \quad (20)$$

- The binding energy per nucleon is then calculated to be

$$e_b = \frac{E_b}{A} \quad MeV/nucleon \quad (21)$$

- Graph of binding energy per nucleon



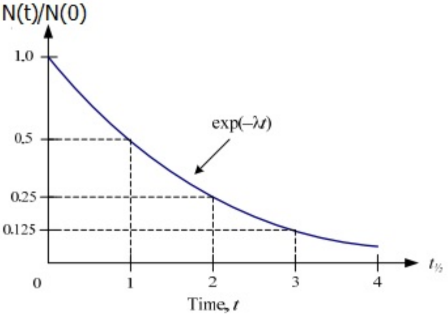
- ^{56}Fe appear at the maximum

Nuclear Decays

- Unstable nuclei will decay into more stable one. Let $N(t)$ be the number of remaining nuclei at any time t can be determined as in the following

$$dN(t) \propto N(t)dt \rightarrow dN(t) = -\lambda N(t)t$$
$$\rightarrow N(t) = N(0)e^{-\lambda t} \tag{22}$$

where λ is a decay constant, in unit of s^{-1}.



- Here we have two time scales of the decay process

$$\text{Half} - \text{life time} \qquad T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda} \tag{23}$$

$$\text{Life time} \qquad \tau = \frac{1}{\lambda} \tag{24}$$

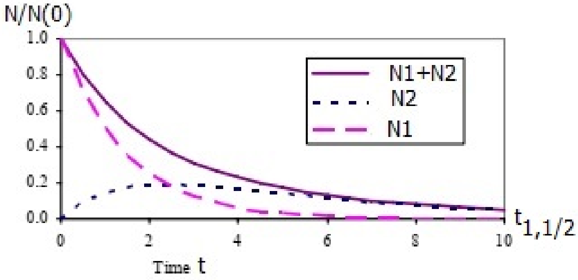
For example of ^{131}I, where $T_{1/2} = 8.0197 days \rightarrow \lambda = 10^{-6} s^{-1}$ and $\tau = 10^6 s = 11.574 days$.

- Let us consider compound decays $N_1 \xrightarrow{\lambda_1} N_2 \xrightarrow{\lambda_2} N_3$, the decay equations are

$$\begin{aligned}
\frac{dN_1}{dt} &= -\lambda_1 N_1 \\
\to N_1(t) &= N_1(0)e^{-\lambda_1 t} && (25) \\
\frac{dN_2}{dt} &= \lambda_1 N_1 - \lambda_2 N_2 \to \frac{dN_2}{dt} + \lambda_2 N_2 = N_1(0)e^{-\lambda_1 t} \\
\to N_2(t) &= N_2(0)e^{-\lambda_2 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0)\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) && (26) \\
\frac{dN_3}{dt} &= \lambda_2 N_2 \\
\to N_3(t) &= N_3(0) + \int_0^t \lambda_2 N_2(t')dt' \\
&= N_3(0) + N_2(0)\left(1 - e^{-\lambda_2 t}\right) \\
&\quad + N_1(0)\left(1 + \frac{\lambda_2}{\lambda_2 - \lambda_1}e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1}e^{-\lambda_2 t}\right) && (27)
\end{aligned}$$

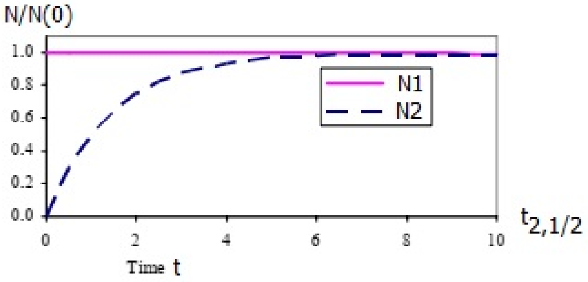
- Note that (4,5,6) is a set of Bateman equations, and their solution are derived by the method of Laplace transformations.
- There are three cases of interest for compound decays
  - *non-equilibrium*, for the case of $\lambda_1 \gg \lambda_2$, with $N_2(0) = 0$, the decay equation becomes

$$N_2(t) = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0) \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) \tag{28}$$



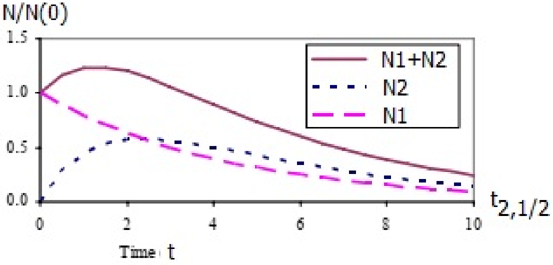
- Three cases of compound decays (cont.)
  - *secular equilibrium*, for the case of $\lambda_1 \ll \lambda_2$, with $N_2(0) = 0$, decay equation becomes

$$N_1(t) \simeq \frac{\lambda_1}{\lambda_2} N_1(0) \left(1 - e^{-\lambda_2 t}\right) \tag{29}$$



- Three cases of compound decays (cont.)
  - *transient equilibrium*, for the case of $\lambda_1 < \lambda_2$, with $N_2(0) = 0$, decay equation becomes

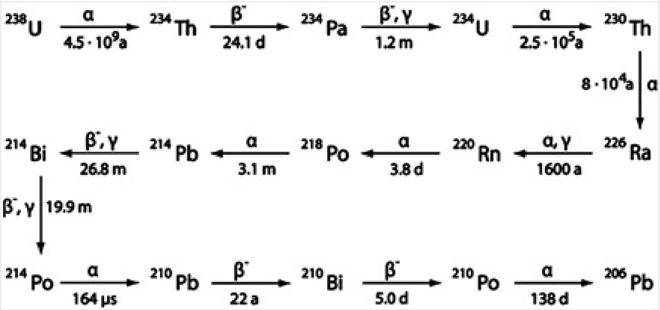
$$N_2(t) \simeq \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0) e^{-\lambda_1 t} \tag{30}$$



- Bateman solution for general compound decays $N_1 \xrightarrow{\lambda_1} N_2 \xrightarrow{\lambda_2} \ldots N_{n-1} \xrightarrow{\lambda_{n-1}} N_n$, with conditions of $N_i(0) = 0$ for $i \geq 2, \ldots, n$ is

$$N_i(t) = \lambda_1 \lambda_2 \ldots \lambda_{n-1} N_1(0) \sum_{j=1}^{n} \frac{e^{-\lambda_j t}}{\prod_{k \neq j, k=1}^{n} (\lambda_k - \lambda_j)} \tag{31}$$

- Example decay series of ^{238}U



Radioactivity

- Decay rate of activity is defined as

$$R(t) = \left| \frac{dN(t)}{dt} \right| = \lambda N(t) \quad (32)$$

In unit of *decay per second* or *Becquerels* (Bq), or *Curie*s (Ci) in which

$$1\ Ci = 3.7 \times 10^10\ Bq \quad (33)$$

- Specific activity is activity per unit mass of radioactive nuclei [Bq/g or Ci/g]

$$SR = \frac{R}{m} = \frac{\lambda N}{m} \quad (34)$$

Radioactive Dating

- The decay constant of a given radio-isotope (unstable nuclei) is not effected by temperature, physical, chemical state, or any other influence of the environment outside the nucleus, so that the decay continue in a predictable rate.
- This makes several types of radioactive dating feasible.
- Starting with the simplest case where there are no daughter atoms present and no mass is lost from the sample, the age can be determined by measuring the relative amounts of the isotopes available at the beginning in the past and the remaining at present.
- Let N_P is a number of "parent nuclei" and N_D is a number of "daughter nuclei", with a single channel decay of the parent, the conservation equation of particles say that

$$N_P(t_0) = N_P(t) + N_D(t), \quad t > t_0 \tag{35}$$

From the decay law

$$N_P(t) = N_P(t_0)e^{-\lambda(t-t_0)} \rightarrow \Delta t = t - t_0 = \frac{1}{\lambda}\ln\left(\frac{N_P(t_0)}{N_P(t)}\right) \tag{36}$$

$$\rightarrow \Delta t = t - t_0 = \frac{1}{\lambda}\ln\left(\frac{R_P(t_0)}{R_P(t)}\right) \tag{37}$$

with $R = \lambda N$ is the activity.

- More complicate situation occur when there are production of daughter nuclei form the other source in the same environment, the particle number conservation becomes

$$N_P(t_0) + N_D(t_0) = N_P(t) + N_D(t), \quad t > t_0 \tag{38}$$

- Fortunately for radioactive dating processes, additional information is available in the form of other isotopes of the daughter elements D' involved in the radioactive process. The particle number conservation equation can be written in term of the ratio as

$$N_P(t_0) + N_D(t_0) \quad = \quad N_P(t) + N_D(t) \qquad t > t \tag{39}$$

In case of D' is not radioactive, we can apply decay law to (18) and we get

$$\frac{N_D(t)}{N_{D'}(t)} = \frac{N_P(t)}{N_{D'}(t)}\left(e^{\lambda \Delta t} - 1\right) + \frac{N_D(t_0)}{N_{D'}(t_0)} \tag{40}$$

We still have many unknowns to solve directly for Δt, but will can determine from graphical method. Let $y = N_D(t)/N_{D'}(t)$ and $x = N_P(t)/N_{D'}(t)$, from (19) we have

$$y = x\left(e^{\lambda \Delta t} - 1\right) + y_0 \to y(x) - \text{graph (isochron)} \tag{41}$$

$$\to \frac{dy}{dx} = e^{\lambda \Delta t} - 1, \quad \text{slope of the graph} \tag{42}$$

Evaluation

$$\Delta t = \frac{1}{\lambda}\left(1 + \frac{dy}{dx}\right) \tag{43}$$

Example of Rb-Sr isochron dating, $T_{1/2}(^{87}Sr) = 48.8 \times 10^9 yrs$ and $dy/dx = 0.0665$, then $\Delta t = 4.53 \times 10^9 yrs$.

(*The data plotted here is from G. W. Wetherill, Ann. Rev. Nucl. Sci. 25,*

Rubidium-Strontium (Rb-Sr) Isochron, to estimate the edge of the Earth

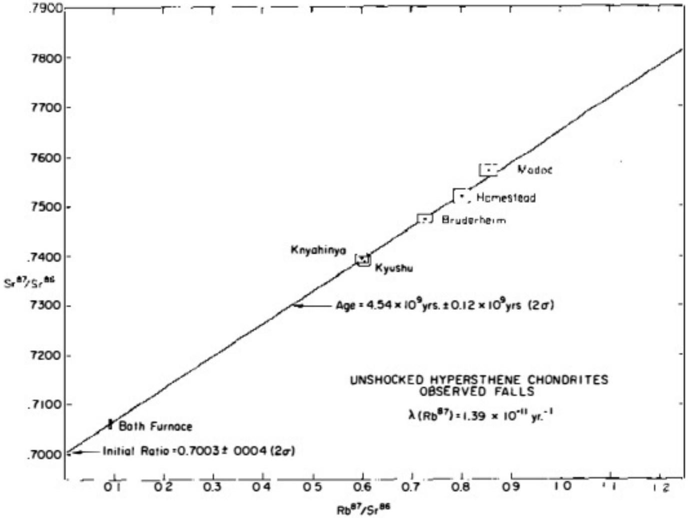


Figure 1 ^{87}Rb-^{87}Sr evolution diagram for six hypersthene chondrite meteorites (93). The data can be interpreted as showing that 4.54 b.y. ago all of these rocks