ICPY473 Nuclear Physics
Wednesday 5, May 2021

4 Nuclear Models II

In order to get more realistic descriptions of nuclear models., we have to apply quantum many-body theory to the nucleus.

4.1 Fermi gas model

Nucleons are fermions, the simplest nuclear many-body model is *Fermi gas model*. Let us determine atomic nucleus as a group of nucleon confined within a rigid cube of volume $V = L^3$. The quantum of nucleon momentum and energy are

$$\vec{k} = (k_x, k_y, k_z),\ k_x = \frac{n_x \pi}{L},\ k_y = \frac{n_y \pi}{L},\ k_z = \frac{n_z \pi}{L} \tag{4.1}$$

$$n_x, n_y, n_z = 1, 2, 3, ... \tag{4.2}$$

$$E_{n_x,n_y,n_z} = \frac{\hbar^2 k^2}{2m_n} = \frac{\hbar^2 \pi^2}{2m_n L^2}(n_x^2 + n_y^2 + n_z^2) \tag{4.3}$$

Let us define *number vector space* as

$$\vec{n} = (n_x, n_y, n_z) \to d^3 n = \frac{L^3}{\pi^3} d^3 k \tag{4.4}$$

$$d^3 \tilde{n} = (2)\frac{1}{8}\frac{L^3}{(\pi\hbar)^3} d^3 p = 2\frac{V}{(2\pi\hbar)^3} d^3 p,\ \vec{p} = \hbar \vec{k} \tag{4.5}$$

where factor (1/8) comes from only positive numbers involve in (4.2), and factor (2) comes from spin-doublet. From system of N-nucleons, we will have

$$\tilde{N} = 2\frac{V}{(2\pi\hbar)^3}\int d^3 p = \frac{8\pi V}{(2\pi\hbar)^3}\int_0^{p_F} p^2 dp = \frac{V p_F^3}{3\pi^2 \hbar^3} \tag{4.6}$$

$$\to p_F = \left(\frac{3\pi^2\hbar^3 \tilde{N}}{V}\right)^{1/3},\ E_F = \frac{p_F^2}{2m_n} = \frac{1}{2m_n}\left(\frac{3\pi^2\hbar^3 \tilde{N}}{V}\right)^{2/3} \tag{4.7}$$

Since the proton and nucleon occupy different potential well, the we have

$$Z = \frac{V (p_F^p)^3}{3\pi^2\hbar^3} \quad \text{and} \quad N = \frac{V (p_F^n)^3}{3\pi^2\hbar^3} \tag{4.8}$$

Since

$$r = r_0 A^{1/3} \to V = \frac{4\pi r^3}{3} = \frac{4\pi r_0^3 A}{3} \tag{4.9}$$

$$\tilde{N} = \frac{4\pi r_0^3 A}{3} \cdot \frac{p_F^3}{3\pi^2\hbar^3} = \frac{4Ar_0^3}{9\pi^2\hbar^3} p_F^3 \to p_F = \left(\frac{9\pi\tilde{N}}{4A}\right)^{1/3} \frac{\hbar}{r_0} \tag{4.10}$$

$$E_F = \frac{p_F^2}{2m_n} \to \tilde{N} = \tag{4.11}$$

In case of $\tilde{N} = N = P = A/2$, we will have

$$p_F = \left(\frac{9\pi}{8}\right)^{1/3} \frac{\hbar}{r_0} \simeq 250MeV/c \text{ and } E_F \simeq 33MeV \tag{4.12}$$

Together with nucleon binding energy, we can estimate the depth of nuclear potential well to be $V_0 \sim -40MeV$.

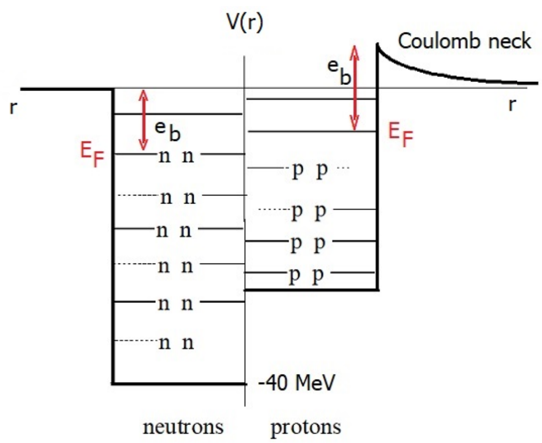

Figure 4.1:

The Coulomb repulsion potential neck is

$$V_C(r) = (Z-1)\frac{k_e e^2}{r} = (Z-1)\frac{\alpha \cdot \hbar c}{r}, \ \alpha = \frac{k_e e^2}{\hbar c} \simeq \frac{1}{137} \tag{4.13}$$

The average energy per nucleon is

$$\langle E \rangle = \frac{\int_0^{E_F} E \frac{dN}{dE} dE}{\int_0^{E_F} \frac{dN}{dE} dE} = \frac{\int_0^{p_F} E \frac{dN}{dp} dp}{\int_0^{p_F} \frac{dN}{dp} dp} = \frac{\int_0^{p_F} \frac{p^2}{2m_n} p^2 dp}{\int_0^{p_F} p^2 dp}$$

$$= \frac{3}{5}\frac{p_F^2}{2m_n} \simeq 20MeV \tag{4.14}$$

where $\frac{dN}{dp} = cont.p^2$ and $EdE = pdp$. The total nuclear (kinetic) energy is then equal to

$$E_{tot}(N,Z) = N\langle E_n\rangle + Z\langle E_p\rangle = \frac{3}{10m_n}\left(N(p_F^n)^2 + Z(p_F^p)^2\right)$$
$$= \frac{3}{10m_n}\frac{\hbar^2}{r_0^2}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N^{5/3}+Z^{5/3}}{A^{2/3}} \tag{4.15}$$

We observe that this nuclear energy get maximized at $N = Z = A/2$, i.e.,

$$E_{tot,max} = \frac{3}{10m_n}\frac{\hbar^2}{r_0^2}\left(\frac{9\pi}{8}\right)^{2/3} A \tag{4.16}$$

Taylor expansion of (4.15) in terms of $(N-Z)^2$ and written as $E_{tot} = E_{tot}^{(0)} + E_{tot}^{(1)} + ...$, we find that (4.16) $= E_{tot}^{(0)}$, and

$$E_{tot}^{(1)} = \frac{3}{10m_n}\frac{\hbar^2}{r_0^2}\left(\frac{9\pi}{8}\right)^{2/3}\frac{5(N-Z)^2}{9A} \tag{4.17}$$

4.2 Liquid drop model

The liquid drop model was first proposed by George Gamow and further developed by Niels Bohr and John Archibald Wheeler. It treats the nucleus as a drop of incompressible fluid of very high density, held together by the nuclear force. The nuclear binding energy is then determined from the energy of a drop of the liquid as

$$E_b(A,Z) = a_V A - a_S A^{2/3} - a_C\frac{Z(Z-1)}{A^{1/3}} - a_A\frac{(N-Z)^2}{A} + \delta(A,Z) \tag{4.18}$$

where

$$\begin{aligned} a_V &= 15.85 MeV,\\ a_S &= 18.34 MeV,\\ a_C &= 0.714 MeV,\\ a_A &= 23.21 MeV\\ a_P &= 12.00 MeV, \end{aligned} \tag{4.19}$$

$$\delta(A,Z) = \begin{cases} +\delta_0, & A, Z - even\\ 0, & A - odd\\ -\delta_0, & A, Z - odd \end{cases}, \quad \delta_0 = \frac{a_P}{A^{1/2}} \tag{4.20}$$

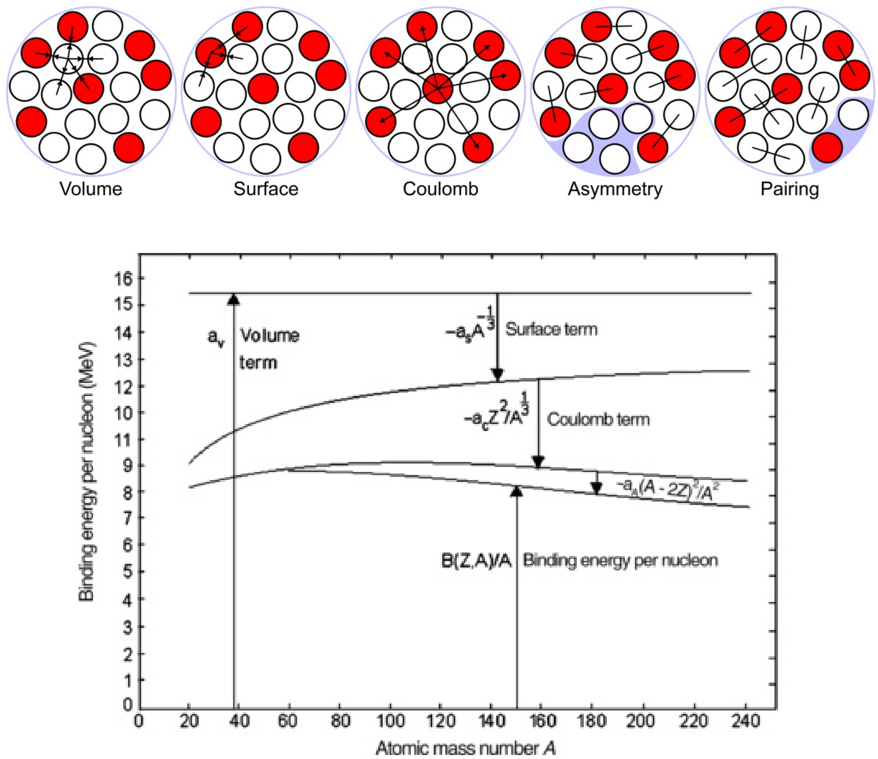

Figure 4.2:

4.3 Collective model

In this model, the nuclear energy is determined from dynamics of the bulk nuclear matter. This dynamics occur in form of *deformed sphere*. From nuclear radius

$$r = r_0 A^{1/3} = R_0 \to R(\theta, \phi; t) = R_0 \left(1 + \sum_{l,m} \alpha_{lm}(t) Y_{lm}(\theta, \phi) \right) \tag{4.21}$$

with

$$
\begin{array}{ll}
l = 0 & (monopole) \\
l = 1 & (dipole) \\
l = 2 & (quadrupole) \\
l = 3 & (octupole) \\
l = 4 & (hexadecupole)
\end{array}
$$

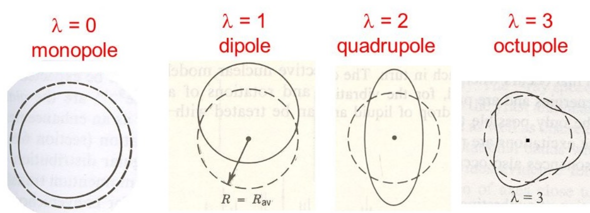

Figure 4.3:

Its dynamics will appear in terms of *rotation* and *vibration* (breathing), for example of rotational energy

$$
E_j^{rot} = \frac{\hbar^2 j(j+1)}{I}, \quad \pi = (-1)^j \ (parity) \tag{4.22}
$$

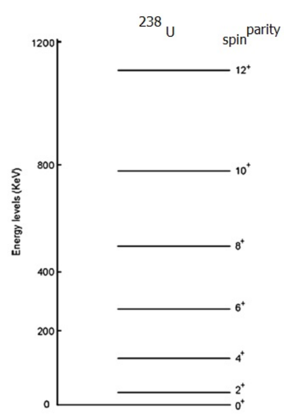

Figure 4.4:

The vibrational energy is

$$E_n^{vib} = (n + 3/2)\hbar\omega_n \tag{4.23}$$

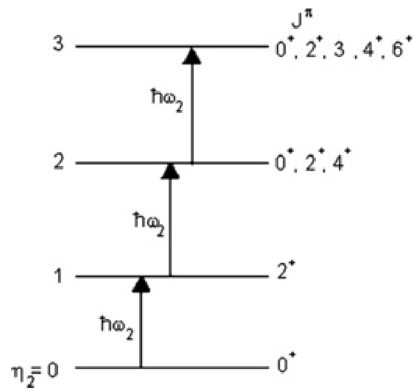

Figure 4.5: