ICPY473 Nuclear Physics
Supplement Document

Two Angular momenta Addition

Let there be $\vec{L}_i,\ i = 1,2$ with $L_i^2|l_i, m_i\rangle = \hbar^2 l_i l_i + 1)|l_i, m_i\rangle, L_{iz}|l_i, m_i\rangle = m_i\hbar|l_i, m_i\rangle$. Determine

$$\vec{J} = \vec{L}_1 + \vec{L}_2 \tag{1}$$

Assume

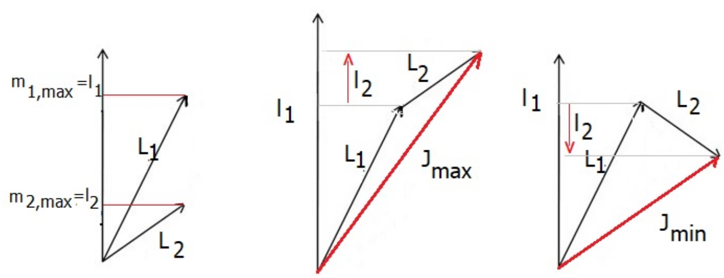
$$J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle,\ J_z|j,m\rangle = \hbar m|j,m\rangle \tag{2}$$

We can see graphically that

$$m = m_1 + m_2 \rightarrow |m_{max}| = j_1 + j_2 \rightarrow j_{max} = l + 1 + l_2 \tag{3}$$

$$\rightarrow |m_{min}| = |j_1 - j_2| \rightarrow j_{min} = |l_1 - l_2| \tag{4}$$

$$\text{Or}\ \ j = l_1 + l_2, l_1 + l_2 - 1, ..., |l_1 - l_2| \geq 0 \tag{5}$$



Since

$$\sum_{m_i} |l_i, m_i\rangle\langle l_i, m_i| = 1$$

Let us define

$$|\{l_1, l_2\}; m_1, m_2\rangle = |l_1, m_1\rangle \otimes |l_2, m_2\rangle \tag{6}$$

$$\sum_{m_1, m_2} |\{l_1, l_2\}; m_1, m_2\rangle\langle\{l_1, l_2\}; m_1, m_2| = 1 \tag{7}$$

$$\rightarrow |j,m\rangle = \sum_{m_1,m_2} |\{l_1, l_2\}; m_1, m_2\rangle \underbrace{\langle\{l_1, l_2\}; m_1, m_2|j,m\rangle}_{=C^{j,m}_{l_1,l_2;m_1,m_2}}$$

$$= \sum_{m_1,m_2} C^{j,m}_{l_1,l_2;m_1,m_2}|\{l_1, l_2\}; m_1, m_2\rangle \tag{8}$$

where $C^{j,m}_{l_1,l_2;m_1,m_2}$ is known as *Clebsch-Gordan coefficients.*

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$

Notation:

		J	J	…
		M	M	…
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	1 +1
+1/2	+1/2	1

m_1	m_2	1 0	0 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1 −1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2 +3/2
+1	+1/2	1

m_1	m_2	3/2 +1/2	1/2 +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2 −1/2	1/2 −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2 −3/2
−1	−1/2	1

2×1

m_1	m_2	3 +3
+2	+1	1

m_1	m_2	3 +2	2 +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3 +1	2 +1	1 +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3 0	2 0	1 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3 −1	2 −1	1 −1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3 −2	2 −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3 −3
−2	−1	1

1×1

m_1	m_2	2 +2
+1	+1	1

m_1	m_2	2 +1	1 +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2 0	1 0	0 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2 −1	1 −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2 −2
−1	−1	1

2×1/2

m_1	m_2	5/2 +5/2
+2	+1/2	1

m_1	m_2	5/2 +3/2	3/2 +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2 +1/2	3/2 +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2 −1/2	3/2 −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2 −3/2	3/2 −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2 −5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2 +2
+3/2	+1/2	1

m_1	m_2	2 +1	1 +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2 0	1 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2 −1	1 −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2 −2
−3/2	−1/2	1

3/2×1

m_1	m_2	5/2 +5/2
+3/2	+1	1

m_1	m_2	5/2 +3/2	3/2 +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2 −3/2	3/2 −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2 −5/2
−3/2	−1	1