ICPY473 Nuclear Physics

Third Trimester, 2021
Udom Robkob, Physics-MUSC
Wednesday 2, June 2021

11 Theory of Nuclear Fusion

The fusion of two nuclei take action of two fundamental forces, long range Coulomb repulsion and short range nuclear strong force attraction. The projectile nucleus have to pass through Coulomb barrier, by quantum tunneling, to hit the target nucleus, with a chance of nuclear cross section.

11.1 Reaction rate

For a well prepared nuclear fusion $X(a,b)Y$, the reaction rate per unit volume will take the form

$$R_{aX} = \frac{1}{1+\delta_{aX}} n_a \cdot n_X \cdot \sigma \cdot v \quad [reaction \cdot s^{-1} \cdot m^{-3}] \tag{11.1}$$

where n_a is projectile nucleus number density, n_X is target nucleus number density, T is Coulomb barrier tunneling coefficient, $\sigma = \sigma_{geo} \cdot T$ is fusion cross section (with T is tunneling coefficient through Coulomb barrier) and v is relative velocity. Note that $\delta_{aX} = 1$ is a correction factor when the projectile and target nuclei are the same type.

Note that $R_{aX} = P_Y$ is the same as production rate of the nucleus Y.

The destruction rate of the nucleus X can be determined to be

$$D_X = -\frac{dn_x}{dt} - (1+\delta_{aX})R_{aX} = -n_a n_X \sigma v \equiv -\frac{n_X}{\tau_a(X)} \tag{11.2}$$

$$\tau_a(X) = \frac{1}{n_a \sigma v} \tag{11.3}$$

where $\tau_a(X)$ is the destruction time scale of the X nucleus against the reaction with a.

The energy production rate per unit volume can be written in the form

$$E_{X(a,b)Y} = R_{aX} Q_{X(a,b)Y} \quad [MeV/s \cdot m^3] \tag{11.4}$$

11.2 Coulomb barrier tunneling coefficient

The Coulomb barrier is

$$V_C(r) = \frac{Z_a Z_X e^2}{4\pi\epsilon_0 r} = \frac{1.44 Z_a Z_X}{r} \quad [MeV/fm] \tag{11.5}$$

after we have used the fact that $e = 1.6 \times 10^{-19} C, \epsilon_0 = 8.85 \times 10^{-12} C \cdot N^{-1} \cdot m^{-2}$, all proportional constant appear to be $2.3 \times 10^{-13} J$ or $1.44 MeV$, when r is measured in fm.

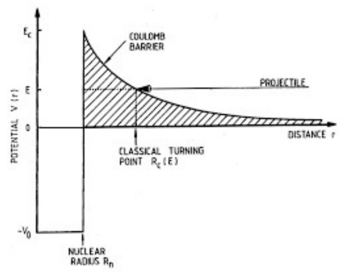

Figure 11.1: Coulomb barrier.

We ever have been derived an expression of quantum tunneling coefficient, from WKB approximation, in alpha decay mechanism. Its expression can be adapted to nuclear fusion in the form

$$T \simeq \exp\left[-\frac{2\pi}{\hbar}\sqrt{\frac{m}{2E}}\frac{Z_a Z_X e^2}{4\pi\epsilon_0}\right] = \exp\left[-\frac{2\pi(1.44)Z_a Z_X}{\hbar v}\right] \equiv e^{-2\pi\eta} \quad (11.6)$$

where $v = \sqrt{2E/m}$ is relative velocity and $\eta = 1.44 Z_a Z_X / \hbar v$ is known as *Sommerfeld parameter*

11.3 Direct nuclear reaction cross section

From the geometry of nuclear reaction below

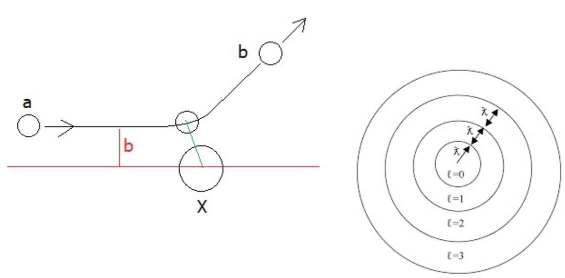

Figure 11.2: Direct nuclear reaction cross section.

Let us determine the angular momentum of the projectile nucleus

$$\text{Classically} \qquad l = bp \tag{11.7}$$

$$\text{Quantum} - \text{mechanically} \qquad l\hbar = b_l p = \frac{\hbar b_l}{\tilde{\lambda}},\ \tilde{\lambda} = \frac{\lambda}{2\pi} \tag{11.8}$$

where $p = h/\lambda$ is the de Broglie particle-wave. From quantum mechanics, l can take integer values, while b_l cannot be measured exactly, according to Heisenberg uncertainty principle.

The projection with and an energy E and the angular momentum quantum l can be characterized by an impact parameter b_l, with the cross section seen by the projectile as a ring surface of

$$\sigma_l(E) = \pi\left[(l+1)^2 - l^2\right]\tilde{\lambda}^2 = (2l+1)\pi\tilde{\lambda}^2 \tag{11.9}$$

With the fact that we have l_{max} at $b_{l_{max}} = R_n$, the radius of nuclear potential well, then

$$\sigma(E) = \sum_{l=0}^{l_{max}} \sigma_l(E) = \pi\tilde{\lambda}^2 \sum_{l=0}^{l_{max}} (2l+1) = \pi\tilde{\lambda}^2(l_{max}+1)^2 \tag{11.10}$$

$$= \pi(R_n + \tilde{\lambda})^2 \tag{11.11}$$

And we will have, by definition,

$$\sigma(E) = \begin{cases} \pi R_n^2, & High\ E:\ \tilde{\lambda} \ll R_n, T(E) \sim 1 \\ \pi\tilde{\lambda}^2 T(E), & Low\ E:\ \tilde{\lambda} \gg R_n \end{cases} \tag{11.12}$$

Note that at low energy

$$\sigma(E) = \pi\tilde{\lambda}^2 T(E) = \frac{\pi\hbar^2}{p^2} T(E) = \frac{\pi\hbar^2}{2mE} e^{-bE^{-1/2}} \tag{11.13}$$

$$\rightarrow\ \sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}} \tag{11.14}$$

where $S(E)$ is known as *astrophysical S-factor*, represents intrinsic nuclear part of the reaction probability, and

$$b = 2\pi\eta E^{1/2} = 31.28 Z_a Z_x \mu^{1/2}\ keV^{1/2}$$

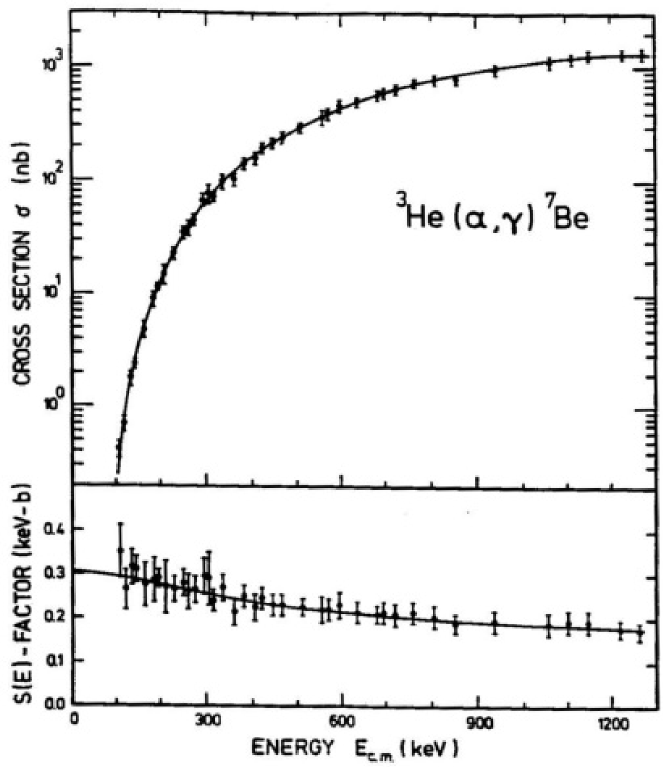

Figure 11.3: The fusion reaction cross section.

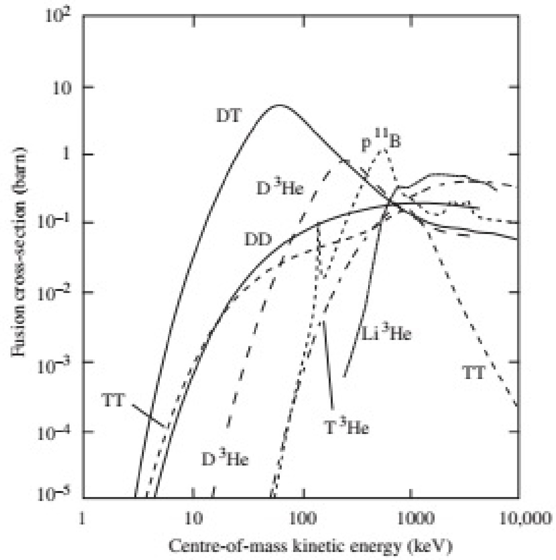

Figure 11.4: Some important fusion reactions cross section.

11.4 Thermal averaged fusion reaction rate

Fusion reaction in stellar plasma.
Thermal distribution function (Boltzmann) of particle at energy E in an environment of temperature T is

$$P(E,T) = 8\pi\sqrt{\frac{\mu}{(2\pi k_B T)^3}} E e^{-E/k_B T} \tag{11.15}$$

where μ is a reduced mass and $E = \frac{1}{2}\mu v^2$. So that the thermal averaged of fusion reaction rate will be

$$\langle R_{aX} \rangle = \frac{1}{1+\delta_{aX}} n_a \cdot n_X \cdot \langle \sigma v \rangle \tag{11.16}$$

$$\to \langle \sigma v \rangle = 8\pi\sqrt{\frac{\mu}{(2\pi k_B T)^3}} \int_0^\infty \sigma(E) E e^{-E/k_B T} dE$$

$$= 8\pi\sqrt{\frac{\mu}{(2\pi k_B T)^3}} \int_0^\infty S(E) e^{-E/k_B T - n/\sqrt{E}} dE \tag{11.17}$$

We observe the *Gamow peak* at

$$e^{-E/k_BT-b/\sqrt{E}} \to \frac{d}{dE}\left(e^{-E/k_BT-b/\sqrt{E}}\right)_{E_0} = 0 \quad (11.18)$$

$$\text{The peak at : } \to E_0 = \left(\frac{bk_BT}{2}\right)^{2/3} = 1.22(Z_a^2 Z_X^2 \mu T_6^2)^{1/3}\ keV \quad (11.19)$$

where T_6 means temperature in million Kelvin.

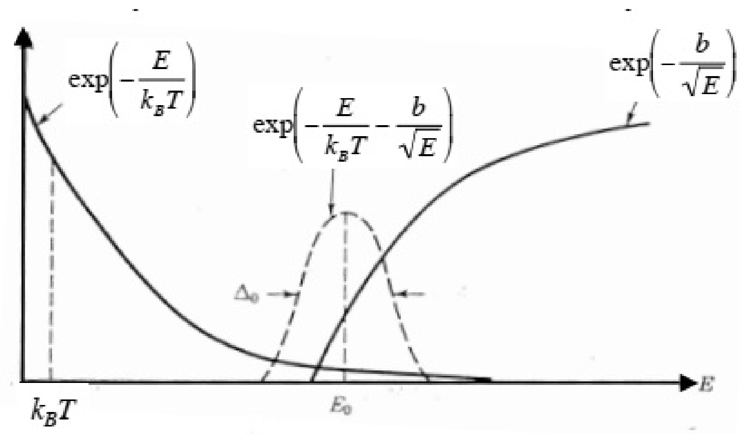

Figure 11.5: The Gamow peak.

We can estimate the *width* Δ of Gamow peak by

$$e^{-E/k_BT-b/\sqrt{E}} \simeq I_{max} e^{-\frac{(E-E_0)^2}{\Delta/2}} \quad (11.20)$$

$$\to I_{max} = \exp\left\{-\frac{E_0}{k_BT} - \frac{b}{\sqrt{E_0}}\right\} = e^{-3E_0/k_BT} \quad (11.21)$$

$$= e^{-42.46(Z_a^2 Z_X^2 \mu/T_6^2)^{1/3}} \quad (11.22)$$

Back insertion of (11.21) into (11.20), and determine the second derivative of both sides at $E = E_0$, at the peak. We observe that

$$\frac{3b}{4}E_0^{-5/2} = \frac{2}{(\Delta/2)^2} \quad (11.23)$$

$$\frac{1}{k_BT} = \frac{1}{2}bE_0^{-3/2} \to \Delta = \frac{4}{\sqrt{3}}\sqrt{k_BTE_0} \quad (11.24)$$

$$= 0.75(Z_a^2 Z_X^2 \mu/T_6^5)^{1/6}\ keV \quad (11.25)$$

For example of ${}^{12}C(p,\gamma)^{13}N$, at $T = 30MK(k_BT = 2.6keV)$, we find that $E_0 = 38keV, \Delta = 23keV$

$(T = 1.5 \cdot 10^7 K)$	$\Delta/2$	$\Delta \cdot I_{\max}$
p + p	3.2 keV	$7.0 \cdot 10^{-6}$ keV
p + ^{14}N	6.8 keV	$2.5 \cdot 10^{-26}$ keV
α + ^{12}C	9.8 keV	$5.9 \cdot 10^{-56}$ keV
^{16}O + ^{16}O	20.2 keV	$2.5 \cdot 10^{-237}$ keV

Figure 11.6: Some Gamow peak data.

Problems 11

11.1. (a) Compute the Coulomb threshold energy for the $p(p, e^+\nu)D$ fusion (in eV) Using the fact that $kee^2 = 1.44 \times 10^{-10} eV$.

(b) Compute the Coulomb threshold energy of the reaction $^{12}C(\alpha, \gamma)^{16}O$.

(c) Compare the results in (a) and (b). To which temperatures (in K) to these energies correspond?.

11.2. Compute the energy of the Gamow peak of $p(p, e^+\nu)D$ reaction at $T = 10^7 K$.

11.3. Assume the temperature at the center of the Sun is $10^7 K$, compute the reaction rate of $p(p, e^+\nu)D$ in this region.