Lecture 13 Relativistic Kinematics

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Special Relativity: SR

- Einstein's theory of special relativity, the two postulates:
  - Newton's theory still valid at high speed relative motion
  - All observers observe the same light speed c
- Two effects on our observation:
  - time dilation
    $$\Delta t = \gamma \Delta t_0,\ \gamma = \frac{1}{\sqrt{1-\beta^2}},\ \beta = \frac{v}{c}$$
    where γ is called *Lorentz factor*, and Δt_0 is called *proper time*
  - length contraction
    $$\Delta l = \frac{1}{\gamma}\Delta l_0$$
    where Δl_0 is called *proper length*

- Mathematics of SR: four dimensional Minkowski space $\mathcal{M}^4$

$$x \in \mathcal{M}^4,\ x = x^\mu e_\mu,\ x^\mu = (x^0, x^1, x^2, x^3) \tag{1}$$

- Lorentz transformation $x \to x'$, with relative motion in 3-direction:

$$x'^0 = \gamma(x^0 - \beta x^3),\ x'^3 = \gamma(x^3 - \beta x^0),\ x'^1 = x^1,\ x'^2 = x^2 \tag{2}$$

  with $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$, we find that
  - $x^3 = 0 \to x^0 = t_0 \to x'^0 = t' = \gamma t_0$, time dilation
  - $x^0 = 0 \to x^3 \neq l_0,\ x'^0 = l_0 \to x^3 = l = \frac{1}{\gamma} l_0$, length contraction (results from simultaneity)

- Lorentz invariant quantity:

$$x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^2)_{rest} = \tau^2$$

  where $\tau = t_0$ proper time

- Relativistic kinematics:
  - 4-position
$$x^\mu = (ct, \vec{x}),\ x^2 = \tau^2$$
  - 4-velocity
$$v^\mu = \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{dt} = \gamma(c, \vec{v}),\ v^2 = c^2$$
  - 4-momentum
$$p^\mu = mv^\mu = (\gamma mc, \gamma m\vec{v}) = (E/c, \vec{p})$$
$$p^2 = m^2c^2 = E^2/c^2 - \vec{p}^2 \to E^2 = \vec{p}^2c^2 + m^2c^4$$
    We have used the fact that $E = \gamma mc^2$, and we observe relativistic momentum $\vec{p} = \gamma m\vec{v}$.
- Some useful relations
$$\gamma = \frac{E}{mc^2},\ \vec{\beta} = \frac{\vec{v}}{c} = \frac{\vec{p}}{E}$$

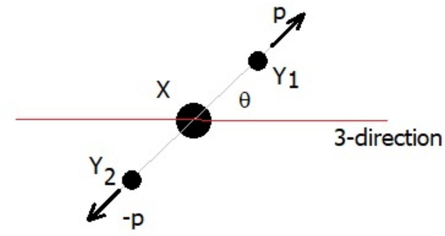
Decays of Unstable Particles

- One-to-two particle decay, with decay equation

$$X \to Y_1 + Y_2$$

- Conserve energy-momentum $p^\mu = (p_1 + p_2)^\mu$
- In the rest (CM) frame of X-particle

$$p_x^\mu = (M_x, 0, 0, 0), \;\; p_1^\mu = (E_1, \vec{p}), \;\; p_2^\mu = (E_2, -\vec{p}) \tag{3}$$



- From energy-momentum conservation

$$M_x = E_1 + E_2 = \sqrt{m_1^2 - p^2} + \sqrt{m_2^2 - p^2} \tag{4}$$

- Square both sides two times, and solve for p, we get

$$p = \frac{1}{2M}\sqrt{(M^2 - (m_1 + m_2)^2)(M^2 - (m_1 - m_2)^2)} \tag{5}$$

This shows that the decay can occur *iff* $M > m_1 + m_2$

- Use (5) in (4) to solve for E_1, E_2, we get

$$E_1 = \frac{1}{2M}(M^2 + m_1^2 - m_2^2) \tag{6}$$

$$E_2 = \frac{1}{2M}(M^2 - m_1^2 + m_2^2) \tag{7}$$

- For special case of $m_1 = m_2 = m$, we have

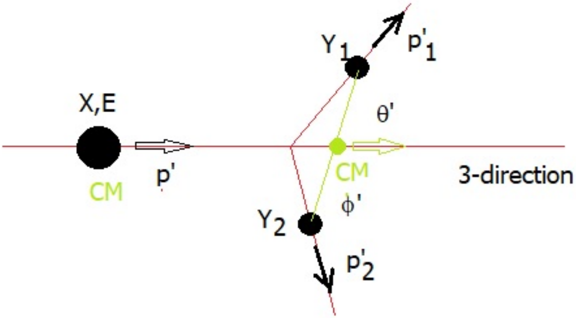
$$\rightarrow p = \frac{1}{2M}\sqrt{M^2 - 4m^2}, E_1 = E_2 = \frac{1}{2}M$$

- For example of a decay $K^0 \to \pi^+ + \pi^-$

$$M(K^0) = 500 MeV, m(\pi^{\pm}) = 140 MeV$$

$$\to p = 207 MeV/c, \;\; E = 250 MeV$$

- In the LAB frame, X-particle may comes in with some energy E' and momentum p' in 3-direction.



- All information in CM-frame will transferred to LAB-frame by Lorentz transformations, i.e., Y_1 particle informations
  - CM-frame:

$$p_1^\mu = (E_1, p_{1\perp},\ p_{1||}), p_{1\perp} = p\sin\theta,\ p_{1||} = p\cos\theta$$

  - LAB-frame:

$$p_1'^{\,\mu} = (E_1', p_{1\perp}', p_{1||}'),\ p_{1\perp}' = p_1'\sin\theta',\ p_{1||}' = p_1'\cos\theta'$$

  - Lorentz factor (determined from X particle):

$$p'^{\,\mu} = (E', 0, 0, p'),\ \gamma = \frac{E'}{M},\ \beta = \frac{p'}{E'}$$

  - LT (c=1)

$$E_1' = \gamma(E_1 + \beta p_{1||}),\ p_{1||}' = \gamma(p_{1||} + \beta E_1),\ p_{1\perp}' = p_{1\perp}$$

$$\frac{p_{1\perp}'}{p_{1||}'} = \tan\theta' = \frac{p_1\sin\theta}{\gamma p_1\cos\theta + \beta E_1} = \frac{\sin\theta}{\gamma\cos\theta + \beta/\beta_1},\ \beta_1 = \frac{p_1}{E_1}$$

- Form an example of a decay $K^0 \rightarrow \pi^+ + \pi^-$, let $\theta = \pi/2$ in CM-frame, then we have

$$p_{1||} = 0, p_{1\perp} = p_1 = 207 MeV/c, \; E_1 = 250 MeV$$

- Let $E'(K^0) = 1 GeV$ in the LAB frame, so that

$$p' = 866 MeV/c, \; \gamma = \frac{1000}{500} = 2, \; \beta = \frac{866}{1000} = 0.866$$

- Apply LT for Y_1 particle

$$E_1' = \gamma(E_1 + \beta p_{1||}) = 2(250) = 500 MeV$$

$$p_1' \perp = \gamma(p_{1||} + \beta E_1) = (2)(0.866)(250) = 433 MeV/c$$

$$p_{1\perp}' = p_{1\perp} = 207 MeV/c \rightarrow p_1' = 479 MeV/c$$

$$\tan\theta' = \frac{\beta_1}{\beta} = 0.956 \rightarrow \theta' = 43.7^0$$

Decay Rate and Lifetime

- From Fermi's golden rule ($\hbar = 1$)

$$W_{fi} = 2\pi|V_{fi}|^2\rho(E_f) \tag{8}$$

where V_{fi} is interaction matrix, which is not Lorentz invariant quantity, $\rho(E_f)$ is density of final state, i.e.,

$$\rho(E_f) = \left(\frac{dn}{dE}\right)_{E_f} = \int \frac{dn}{dE}\delta(E_i - E)dE,\ E_i = E_f \tag{9}$$

$$\rightarrow W_{fi} = 2\pi \int |V_{fi}|^2\delta(E_i - E)dn \tag{10}$$

- From basic QM, we know that

$$dn = \frac{d^3\vec{p}}{(2\pi)^3},\ \text{with the fact that } E = E_1 + E_2 \tag{11}$$

$$\rightarrow W_{fi} = 2\pi \int |V_{fi}|^2\delta(E_i - E_1 - E_2)\frac{d^3\vec{p}_1}{(2\pi)^3} \tag{12}$$

- From momentum conservation $\vec{p}_i = \vec{p}_1 + \vec{p}_2$, we insert this into (12) as a constraint condition as

$$W_{fi} = (2\pi)^4 \int |V_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^{(3)}(\vec{p}_i - \vec{P}_1 - \vec{P}_2)\frac{d^3\vec{P}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3} \tag{13}$$

- Now replace V_{fi} with the Lorentz invariant one, using the fact that

$$\langle\psi_{NR}|\psi_{NR}\rangle = 1 \text{ but } \langle\psi'_R|\psi'_R\rangle = 2E \tag{14}$$

$$\rightarrow |\psi'_R\rangle = \sqrt{2E}|\psi_{NR}\rangle, \text{ and } M_{fi} = \langle\psi'_1\psi'_2|V|\psi'_i\rangle = \sqrt{2E_1 2E_2 2E_i}V_{fi} \tag{15}$$

Apply into (13), we have

$$W_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^{(3)}(\vec{p}_1 - \vec{P}_1 - \vec{P}_2) \times \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \tag{16}$$

- Note that

$$\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} = LIPS$$

where LIPS = *Lorentz invariant phase space integral* of the final states.

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where LIPS = *Lorentz invariant phase space integral* of the final states.
- Let us determine W_{fi} in (16) in the CM-frame, here we have
$$p_i^\mu = (M, 0, 0, 0),\ \ p_1^\mu = (E_1, \vec{p}),\ \ p_2^\mu = (E_2, -\vec{p})$$
So that
$$W_{fi} = \frac{1}{8\pi^2 M}\int |M_{fi}|^2 \delta(M - E_1 - E_2)\delta^{(3)}(\vec{p}_1 + \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2} \quad (17)$$
$$= \frac{1}{32\pi^2 M}\int |M_{fi}|^2 \delta(M - E_1 - E_2)\frac{d^3\vec{p}_1}{E_1 E_2} \quad (18)$$

- Using the fact that $E = \sqrt{p^2 + m^2}$, and rewrite $d^3\vec{p}$ is spherical coordinate, we have from (18)

$$W_{fi} = \frac{1}{32\pi^2 M} \int |M_{fi}|^2 \delta\left(M - \sqrt{p^2 + m_1^2} - \sqrt{p^2 + m_2^2}\right) \times \frac{p^2 dp d\Omega}{\sqrt{p^2 + m_1^2}\sqrt{p^2 + m_2^2}} \tag{19}$$

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- Let us do calculation of the integral

$$I = \int I(p) g(p) \delta(f(p)) dp$$

where

$$g(p) = \frac{p^2}{E_1 E_2}, \; f(p) = M - \sqrt{p^2 + m_1^2} - \sqrt{p^2 + m_2^2}$$

- Let $f(p^*) = 0$, we can observe that

$$p^* = \frac{1}{2M}\sqrt{(M^2 - (m+1+m_2)^2)(M^2 - (m_1 - m_2)^2)}$$

and using the identity

$$\delta(f(p)) = \frac{\delta(p - p^*)}{|f'(p^*)|}, \quad |f'(p^*)| = p^* \frac{E_1 + E_2}{E_1 E_2}$$

$$\rightarrow I = I(p^*)\frac{p^*}{E_1 + E_2} = I(p^*)\frac{p^*}{M}$$

- Apply to (19), we get

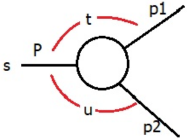
$$W_{fi} = \frac{p^*}{32\pi^2 M^2} \int |M_{fi}(p^*)|^2 d\Omega \equiv \frac{1}{\tau} \tag{20}$$

where τ is a lifetime of the decay.

Mandelstam Variables

- From the decay $X \to Y_1 + Y_2$, with the conserve energy-momentum $P^\mu = (p_1 + p_2)^\mu$, the Mandelstam variables are defined to be

$$s = P^2 = (p_1 + p_2)^2,\ t = (P - p_1)^2 = p_2^2,\ u = (P - p_2)^2 = p_1^2 \quad (21)$$



- IN the CM-frame, we observe that

$$s = \sqrt{M},\ p = p^* = \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{2\sqrt{s}} \quad (22)$$

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yx \quad (23)$$

It is called *Stuckelberg function*.

- Finally we have

$$W_{fi} = \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{64\pi^2 M^3} \int |M_{fi}|^2 d\Omega \tag{24}$$

$$= \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{16\pi^2 M^3} |M_{fi}|^2 = \frac{1}{\tau} \tag{25}$$

Note that $|M_{fi}|^2$ always reported for some particular decay in the *pdg,lbl.gov* data base, as a function of (s, t, u)