Oscillations

with Malcolm John

These lectures explore how the weak force gives rise to flavour oscillations and *CP* violation. Flavour oscillation means a reversible transmutation from one flavour to another. This phenomenon is observed within all neutral meson systems ($K^0\overline{K}^0$, $D^0\overline{D}^0$, $B^0\overline{B}^0$, $B_s\overline{B}_s$) and amongst the three neutrino species (ν_e, ν_μ, ν_τ). *CP*-violation is mathematical terminology for a difference between matter and antimatter. It is observed only in a handful of rare kaon and *B*-meson decays. The search for *CP*-violation in neutrinos is a major topic for the next generation of neutrino experiments.

1 Neutral kaon mixing

Fig. 1 show early proof of neutral kaon mixing from a bubble chamber image. The strangeness of the neutral meson appears to change during the flight between creation and decay. This statement is under the [correct] assumption that the creation of the neutral kaon is via the strong force, which is flavour conserving. The transition,

$$K^+ + p^+ \to K^0 + p^+ + \pi^+ , \qquad |K^+\rangle = |\bar{s}u\rangle, \;\; |K^0\rangle = |\bar{s}d\rangle,$$

conserves *antistrangeness*. Conversely, the baryon-number-conserving transition,

$$\overline{K}^0 + p^+ \to \Lambda^0 + \pi^+ + \pi^0 , \qquad |\overline{K}^0\rangle = |s\bar{d}\rangle, \;\; |\Lambda^0\rangle = |sud\rangle,$$

conserves *strangeness*. The neutral kaon has transmuted from $|\bar{s}d\rangle$ to $|s\bar{d}\rangle$ in a few centimetres. For this to occur, the K^0 and $\overline{K}^0$, cannot be the "mass eigenstates" of the Hamiltonian, those that propagate in time. The mass eigenstates (i.e. the *K*-short, K^0_{S} and *K*-long, K^0_{L} as we know them today) are linear superpositions of the flavour eigenstates (states of definite flavour) $\overline{K}^0$ and K^0 and hence have access to both in interactions.

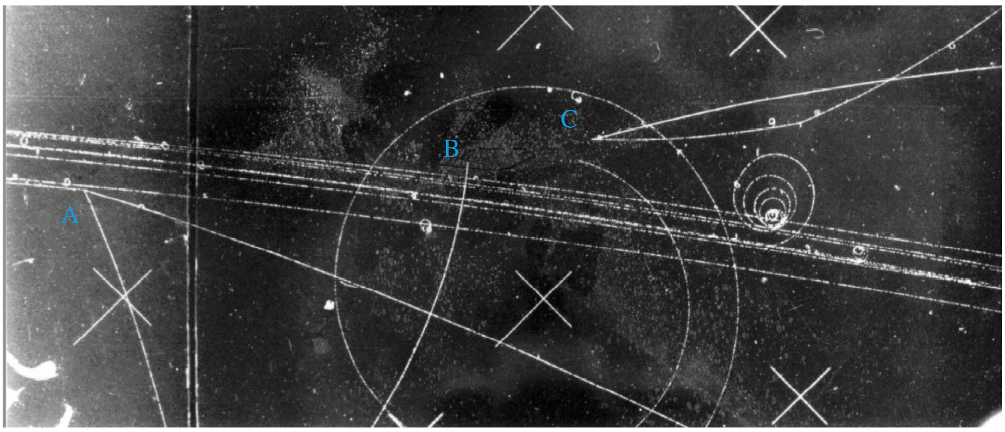


Figure 1: a K^+ beam enters the left edge of a bubble chamber and scatters off a proton at A. A pion and a neutral kaon is formed. The kaon flies to the right without leaving an trace and the recoiling proton and pion move to the bottom of the image. At B, the neutral kaon scatters off another proton to form a Λ^0 baryon with an associated π^+, as well as an untraced π^0. At C, the Λ^0 baryon decays into a proton and a π^- in a characteristic "V" decay.

1.1 Formalism

The time evolution of one neutral meson, in its rest frame, may be written as[1],

$$|K^0(t)\rangle = e^{-iMt}e^{-\Gamma t/2}|K^0\rangle$$

where the first exponential is a plane-wave solution, $\exp(\frac{i}{\hbar}(\mathbf{p}.\mathbf{x} - Et))$ for a state with energy $E = M$ in its rest frame, $|\mathbf{p}| = 0$, with $\hbar = 1$. The second term describes the exponential decay for a state with proper lifetime τ (i.e. width $\Gamma = \hbar/\tau$, again with $\hbar = 1$) such that,

$$|\langle K^0|K^0(t)\rangle|^2 \propto e^{-t/\tau}$$

Generalise to a two state system with 2×2 matrices, $\mathbf{M}$ and $\mathbf{\Gamma}$ encoding the time evolution,

$$\begin{pmatrix} |K^0(t)\rangle \\ |\overline{K}^0(t)\rangle \end{pmatrix} = \varSigma \begin{pmatrix} |K^0\rangle \\ |\overline{K}^0\rangle \end{pmatrix} \qquad \text{where } \varSigma = e^{-i\mathbf{M}t - \mathbf{\Gamma}t/2}\,, \tag{1}$$

where the presence (absence) of the bar above the $K(t)$ shows the flavour of the state at $t = 0$: pure $|\overline{K}^0\rangle$ $\big($or pure $|K^0\rangle\big)$. The states $|K^0\rangle$ and $|\overline{K}^0\rangle$ are of well defined flavour so it is in this basis that it is appropriate to discuss their interaction by the weak force. Thus they are the *weak* eigenstates, as opposed to the *mass* eigenstates of the Hamiltonian. Apply Schrödinger's equation, $i\, d\psi/dt = \mathrm{H}\psi$, identifies the Hamiltonian.

$$\frac{d}{dt}\begin{pmatrix} |K^0\rangle \\ |\overline{K}^0\rangle \end{pmatrix} = \mathbf{H} \begin{pmatrix} |K^0\rangle \\ |\overline{K}^0\rangle \end{pmatrix} \qquad \text{with} \qquad \mathbf{H} = \mathbf{M} - \frac{i}{2}\mathbf{\Gamma}. \tag{2}$$

Any matrix, $\mathbf{A}$ can be decomposed in the form $\mathbf{B} + i\,\mathbf{C}$ where $\mathbf{B}$ and $\mathbf{C}$ are hermitian because,

$$\begin{aligned} \mathbf{A} &= \frac{\mathbf{A}}{2} + \frac{\mathbf{A}}{2} + \frac{\mathbf{A}^\dagger}{2} - \frac{\mathbf{A}^\dagger}{2} \\ &= \underbrace{\frac{1}{2}\left(\mathbf{A} + \mathbf{A}^\dagger\right)}_{=\,\mathbf{B}\ \ (=\,\mathbf{B}^\dagger)} + \underbrace{\frac{1}{2}\left(\mathbf{A} - \mathbf{A}^\dagger\right)}_{=\,i\mathbf{C}\ \ (=\,-i\mathbf{C}^\dagger)} \qquad \text{by inspection.} \end{aligned}$$

So $\mathbf{M}$ and $\mathbf{\Gamma}$ are hermitian matrices: $M_{21} = M^*_{12}, \Gamma_{21} = \Gamma^*_{12}$. This also implies observable quantities because the eigenvalues of hermitian matrices are real. Last, we impose CPT-invariance (identical mass and lifetimes of the K^0 and $\overline{K}^0$), $M_{11} = M_{22} = M, \Gamma_{11} = \Gamma_{22} = \Gamma$. So the hamiltonian of the meson-antimeson system becomes,

$$\mathbf{H} = \begin{pmatrix} M & M_{12} \\ M^*_{12} & M \end{pmatrix} - \frac{i}{2}\begin{pmatrix} \Gamma & \Gamma_{12} \\ \Gamma^*_{12} & \Gamma \end{pmatrix} \tag{3}$$

The off-diagonal terms describe the transitions [mixing] between the meson and antimeson.

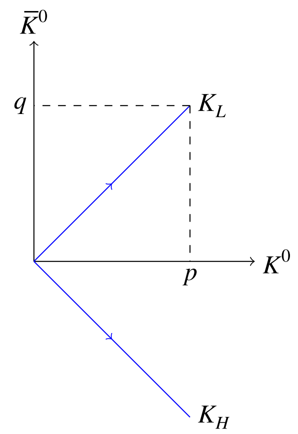
Returning to the wavefunctions, the eigenstates of the hamiltonian propagate in time with defined mass and lifetimes. We label the two mass eigenstates *Heavy*, *Light* and describe them as a combination of the flavour eigenstates,

$$\begin{aligned} |K_H\rangle &= p\,|K^0\rangle - q\,|\overline{K}^0\rangle \\ |K_L\rangle &= p\,|K^0\rangle + q\,|\overline{K}^0\rangle\,, \end{aligned} \tag{4}$$

where $p^2 + q^2 = 1$. The "$p + q,\ p - q$" form is required by orthogonality. Inverting this transformation, gives

$$|K^0\rangle = \tfrac{1}{2p}\left(|K_L\rangle + |K_H\rangle\right) \tag{5}$$

$$|\overline{K}^0\rangle = \tfrac{1}{2q}\left(|K_L\rangle - |K_H\rangle\right)\,. \tag{6}$$



[1] This simplification of time-dependent perturbation theory is known as the *Wigner-Weisskopf* approximation.

Eq. 1 describes the time evolution of the flavour eigenstates and is related to the time evolution of the mass eigenstates by the similarity transformation, $\Sigma = PDP^{-1}$, noting that the the time evolution in the mass basis is diagonal because mass eigenstates propagate in time with definite energy and lifetime.

$$\begin{aligned}\begin{pmatrix}|K(t)\rangle\\|\overline{K}(t)\rangle\end{pmatrix} &= \begin{pmatrix}\frac{1}{2p} & \frac{1}{2p}\\ \frac{1}{2q} & \frac{-1}{2q}\end{pmatrix}\begin{pmatrix}|K_L(t)\rangle\\|K_H(t)\rangle\end{pmatrix}\\ &= \quad '' \quad \begin{pmatrix}e^{-iM_Lt-\Gamma_Lt/2} & 0\\ 0 & e^{-iM_Ht-\Gamma_Ht/2}\end{pmatrix}\begin{pmatrix}|K_L\rangle\\|K_H\rangle\end{pmatrix}\\ &= \quad '' \qquad\qquad '' \qquad \begin{pmatrix}p & q\\ p & -q\end{pmatrix}\begin{pmatrix}|K^0\rangle\\|\overline{K}^0\rangle\end{pmatrix}\end{aligned}$$

In passing, it is noted that from linear algebra, the columns of P are eigenvectors of Σ. Multiplying out,

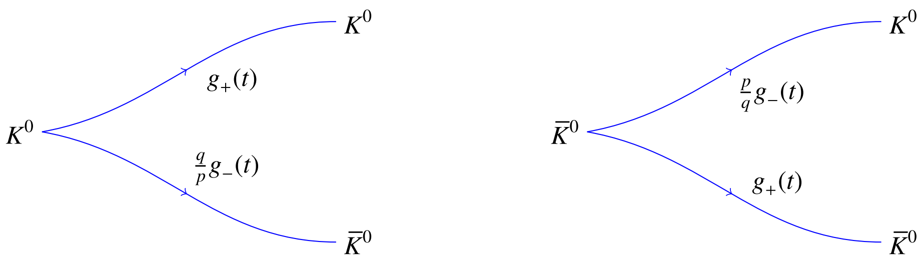
$$\Sigma = \begin{pmatrix}\frac{1}{2p} & \frac{1}{2p}\\ \frac{1}{2q} & \frac{-1}{2q}\end{pmatrix}\begin{pmatrix}e^{-iM_Lt-\Gamma_Lt/2} & 0\\ 0 & e^{-iM_Ht-\Gamma_Ht/2}\end{pmatrix}\begin{pmatrix}p & q\\ p & -q\end{pmatrix} \tag{7}$$

$$= \begin{pmatrix}g_+ & \frac{q}{p}g_-\\ \frac{p}{q}g_- & g_+\end{pmatrix} \quad \text{where} \quad g_\pm = \frac{1}{2}\left[e^{-iM_Lt-\Gamma_Lt/2} \pm e^{-iM_Ht-\Gamma_Ht/2}\right] \tag{8}$$

Hence the time evolution of the decaying state is,

$$\begin{aligned}|K(t)\rangle &= \quad g_+\,|K^0\rangle + \frac{q}{p}g_-\,|\overline{K}^0\rangle\\ |\overline{K}(t)\rangle &= \frac{p}{q}g_-\,|K^0\rangle + \quad g_+\,|\overline{K}^0\rangle\end{aligned} \tag{9}$$

Or illustratively,



From which we note, as a precursor to the discussion on *CP* violation, that a difference in the temporal evolution of neutral-meson mixing can arise if q and p are different,

$$\frac{q}{p} \neq \frac{p}{q} \quad , \quad \left(\frac{q}{p}\right)^2 \neq 1.$$

Applying Schrödinger's equation to the off diagonal part of Eq. 8, and equating to Eq. 3 noting that the time dependance is all contained within the $g_\pm$ factors,

$$\begin{aligned}i\frac{d}{dt}\left(\frac{q}{p}g_-(t)\right) &= i\frac{q}{p}\left[M_L - M_H - \frac{i}{2}\left(\Gamma_L - \Gamma_H\right)\right] = M_{12} - \frac{i}{2}\Gamma_{12}\\ i\frac{d}{dt}\left(\frac{p}{q}g_-(t)\right) &= i\frac{p}{q}\left[M_L - M_H - \frac{i}{2}\left(\Gamma_L - \Gamma_H\right)\right] = M_{12}^* - \frac{i}{2}\Gamma_{12}^*\\ \left(\frac{q}{p}\right)^2 &= \frac{M_{12} - \frac{i}{2}\Gamma_{12}}{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}\end{aligned} \tag{10}$$

Where we remind ourselves that M_{12} and Γ_{12} are the elements of the hamiltonian $\mathbf{H}$ that describe the action of $|\overline{K}^0\rangle$ appearing in the initially $|K^0\rangle_{t=0}$ wavefunction.

1.2 Measuring K^0-$\overline{K}^0$ oscillations

For now, we concentrate on kaon oscillations and ignore *CP* violation, i.e. $p = q$. Neutral kaons can be produced in a state of definite antistrangeness by a strong interaction of negatively-charged pions on a proton target, see Fig. 2(a). By strangeness conservation (strong interaction) and baryon number conservation, it is impossible to make a $\bar{\Lambda}^0$.

In the semileptonic decay, the charge of the muon must be that of the strange quark, see Fig. 2 (b) and (c). Hence by counting the number of $\mu^+\pi^-$ decays versus the number of $\mu^-\pi^+$ decays as a function of decay time, this decay can pick out the proportion of K^0 and $\overline{K}^0$ in the propagating neutral kaon wavefunction.

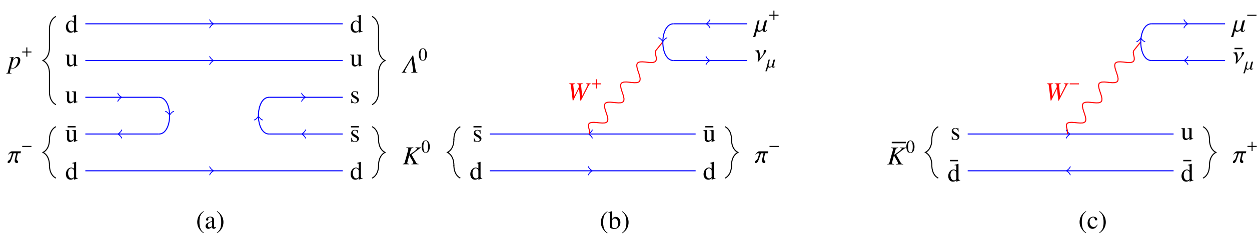


Figure 2: Quark flow diagrams for (a) K^0 production from pion-nucleon scattering (b) semileptonic decay of a K^0 and (c) semleptonic decay of a $\overline{K}^0$. The charge of the muon unabiguously identifies the kaon flavour.

As the system starts with a pure $|K^0\rangle$, Eq. 9 with $p = q$ is appropriate. Writing out completely,

$$\begin{aligned} |K(t)\rangle &= g_+ |K^0\rangle + g_- |\overline{K}^0\rangle \\ &= \tfrac{1}{2}(\underbrace{e^{-iM_Lt-\Gamma_Lt/2}}_{a} + \underbrace{e^{-iM_Ht-\Gamma_Ht/2}}_{b})|K^0\rangle + \tfrac{1}{2}(\underbrace{e^{-iM_Lt-\Gamma_Lt/2}}_{a} - \underbrace{e^{-iM_Ht-\Gamma_Ht/2}}_{b})|\overline{K}^0\rangle \end{aligned} \tag{11}$$

The K^0 ($\overline{K}^0$) intensity is found from the probability of finding a K^0 ($\overline{K}^0$) at time t from this combined wavefunction,

$$\begin{aligned} |\langle K^0|K(t)\rangle|^2 &= \frac{1}{4}(a+b)(a+b)^* \\ &= \frac{1}{4}(aa^* + bb^*) + \frac{1}{4}(ba^* + ab^*) \\ |\langle \overline{K}^0|K(t)\rangle|^2 &= \frac{1}{4}(a-b)(a-b)^* \\ &= \frac{1}{4}(aa^* + bb^*) - \frac{1}{4}(ba^* + ab^*) \end{aligned}$$

Taking first the direct "amplitude-squared" term in each line,

$$\begin{aligned} aa^* + bb^* &= e^{(-iM_L - \frac{\Gamma_L}{2})t} e^{(+iM_L - \frac{\Gamma_L}{2})t} + e^{(-iM_H - \frac{\Gamma_H}{2})t} e^{(+iM_H - \frac{\Gamma_H}{2})t} \\ &= e^{-\Gamma_L t} + e^{-\Gamma_H t} \end{aligned}$$

The interference term reveals a sinusoidal dependence,

$$\begin{aligned} ba^* + ab^* &= e^{(-iM_H - \frac{\Gamma_H}{2})t} e^{(+iM_L - \frac{\Gamma_L}{2})t} + e^{(-iM_L - \frac{\Gamma_L}{2})t} e^{(+iM_H - \frac{\Gamma_H}{2})t} \\ &= e^{-\frac{\Gamma_L+\Gamma_H}{2}t} e^{-i(M_H - M_L)t} + e^{-\frac{\Gamma_L+\Gamma_H}{2}t} e^{i(M_H - M_L)t} \\ &= e^{-\frac{\Gamma_L+\Gamma_H}{2}t} \cdot 2\cos(\Delta M t) \end{aligned}$$

So,

$$\begin{aligned} \mathcal{P}_{K^0}(t) = |\langle K^0|K(t)\rangle|^2 &= \frac{1}{4}\left(e^{-\Gamma_L t} + e^{-\Gamma_H t}\right) + \frac{1}{2} e^{-\frac{\Gamma_L+\Gamma_H}{2}t} \cos(\Delta M t) \\ \mathcal{P}_{\overline{K}^0}(t) = |\langle \overline{K}^0|K(t)\rangle|^2 &= \frac{1}{4}\left(e^{-\Gamma_L t} + e^{-\Gamma_H t}\right) - \frac{1}{2} e^{-\frac{\Gamma_L+\Gamma_H}{2}t} \cos(\Delta M t) \end{aligned} \tag{12}$$

which depends on the K_L and K_H lifetimes and their mass difference[2] in the interference term.

The two kaon mass eigenstates have remarkably different lifetimes and are thus always labelled with reference to that property: the K-short $K^0_{\rm S}$ and the K-long, $K^0_{\rm L}$. The $K^0_{\rm S}$ has lifetime of 89.5 ps whereas the $K^0_{\rm L}$ lives 571 times longer. Experiment determines the $K^0_{\rm S}$ is the lighter-mass eigenstate though there is no fundamental reason why this should be the case.

The distributions of $\mathcal{P}_{K^0}(t)$ and $\mathcal{P}_{\bar{K}^0}(t)$ are shown over 1ns in Fig. 3. As expected for the initially well-defined $|K^0\rangle$, $\mathcal{P}_{K^0}(0) = 1$ and $\mathcal{P}_{\bar{K}^0}(0) = 0$. One nanosecond is over 11 $K^0_{\rm S}$ lifetimes so this component will have fallen to $e^{-\Gamma_L t} \sim 0$, leaving

$$\mathcal{P}_{K^0}(1\text{ ns}) \approx \mathcal{P}_{\bar{K}^0}(1\text{ ns}) \approx \frac{1}{4}e^{-\Gamma_H t}\,.$$

1 ns is about one fiftieth the $K^0_{\rm L}$ lifetime, $\frac{1}{4}\exp^{-\frac{1}{50}} = 0.245$.

Considering the oscillatory term note that $\mathcal{P}_{K^0} = \mathcal{P}_{\bar{K}^0}$ when $\cos(\Delta M t) = 0$. This occurs at $t = \frac{\pi}{2}(\Delta M)^{-1}$, $\frac{3\pi}{2}(\Delta M)^{-1}$ ...
The sketch shown this occurring at 300 ps and 900 ps,

$$\Delta M = \frac{\pi}{2}(300)^{-1} = 5.24\times10^{-3}\,\text{ps}^{-1}\,.$$

Or in eV,

$$\frac{5.24\times10^{-3}}{10^{-12}}[\text{s}^{-1}] \times \frac{\hbar}{e} = 3.45\times10^{-6}[\text{eV}]\,.$$

Incredibly, this is fourteen orders of magnitude smaller than the mean kaon mass, 492 MeV, yet it readily measurable due of the sensitivity of interference phenomena.

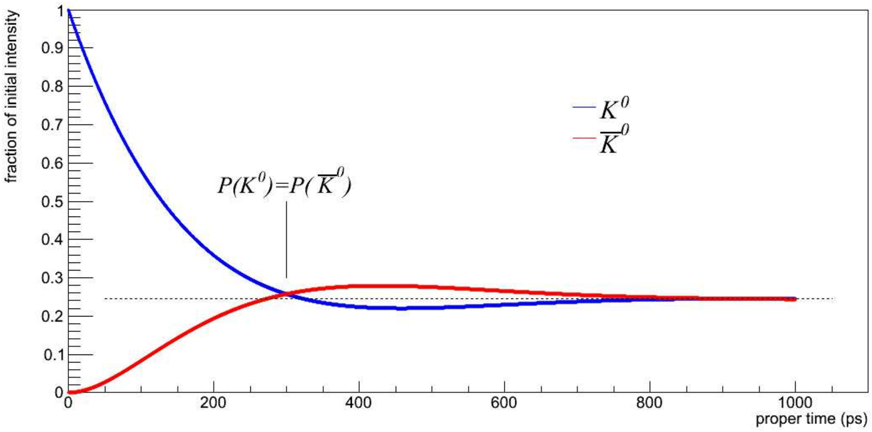


Figure 3: Probabilities of finding each kaon *flavour* eigenstates from from an initially K^0 source.

[2]To reiterate, K_L and K_H are **not** each other's antiparticle. CPT theorem does not apply so they can, and do, have different masses and lifetimes.