SCPY322 Nuclear and Particle Physics
PPI-Relativistic Kinematics
March 26, 2021

1 Special Relativity (SR)

SR has had been originated by Einstein postulates:

- Newton's laws of motion still valid for energetic motion
- All observers observe the same light speed c

This results to our two basic observations

- time dilation $dt = \gamma d\tau$, when $\gamma = (1-\beta^2)^{-1/2}$ is a relativistic (Lorentz) factor and $\beta = u/c$ a relative speed of observation. Note that $d\tau$ is called *proper times*, a time duration have been measured from a clock at rest.
- length contraction $dl = dl_0/\gamma$

Another thing we have to be careful about giving any physical interpretation is *simultaneity* of any two physical events.

2 Lorentz Transformations (LT)

In order to do mathematical calculation about SR, we have to assign the spec-time coordinates (ct, x, y, z) and apply with Lorentz transformations between two reference frames under have relative motion u in z-direction, as

$$ct = \gamma(ct' + uz'/c),\ z = \gamma(z' + ut'),\ x = x',\ y = y' \tag{1}$$

$$ct' = \gamma(t - uz/c),\ z' = \gamma(z - ut),\ x' = x,\ y' = y \tag{2}$$

Exercise 1: Derive expressions of time dilation and length contraction from these LT, using the concept of simultaneity.

3 Covariant Formulation (CF)

To get a unique expression of the LT (and SR) we use the covariant formulation. Let us move onto the four dimensional *Minkowski space* $\mathcal{M}^4$, with coordinate $x^\mu = (x^0, x^1, x^2, x^3)$ and equipped the metric tensor $g_{\mu\nu} = diag.(+1, -1, -1, -1)$. On this space, we calculate the *distance* in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

And ds^2 is invariant on any reference frames on $\mathcal{M}^4$. Note that x^μ is known as a 4-vector.

Apply to LT, we assign with $x^0 = ct, (x^1, x^2, x^3) = \vec{x}$, so that we have a 4-position $x^\mu = (ct, \vec{x})$ and

$$ds^2 = (cdt)^2 - (d\vec{x})^2$$

The invariant property

$$ds_0^2 = (cd\tau)^2 = (cdt)^2 - (d\vec{x})^2 = (cdt)^2\gamma = ds^2$$

We observe time dilation formula $d\tau = \gamma dt$.

4 Relativistic Kinematics (RK)

We start with the 4-position

$$x^\mu = (ct, \vec{x}) \to x^2 = \tau^2$$

Then we define the 4-velocity as

$$v^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{d}{dt}(ct, \vec{x}) = \gamma(c, \vec{v}) \to v^2 = c^2 \tag{3}$$

The 4-momentum is now written in the form

$$p^\mu = mv^\mu = (\gamma mc, \gamma m\vec{v}) \equiv (E/c, \vec{p}) \tag{4}$$

where we have defined the relativistic energy $E = \gamma mc^2$ and relativistic momentum $\vec{p} = \gamma m\vec{v}$, satisfy the relativistic energy-momentum relation

$$E^2 = \vec{p}^2c^2 + m^2c^4 \tag{5}$$

From (4), this leads to $p^2 = m^2c^2$. Two basic of SR for LT of energetic particle can be derived from these formulations are

$$\gamma = \frac{E}{mc^2},\ \vec{\beta} = \frac{\vec{p}}{E}$$

Exercise 2: Note that mc^2 is known as the rest mass energy of a particle of mass m. Let calculate the rest mass energies of and electron (e) and a proton (p), in the unit of eV with appropriate order.

Exercise 3: Calculate the momentum of and electron at energies of (a) 50 keV, (b) 5 MeV, and (c) 500 MeV, in the unit of MeV/c.

Exercise 4: Calculate the velocity of a proton moving with an energy of 7 TeV, within the LHC tunnel.

5 RK of Two-Particle Decays

In a process of heavy particle (mother) of mass M decays into two light particles (daughters) of masses m_1, m_2, thinks we want to know are energies and momenta of the daughters, from the known mother. For simplicity, let us working in the unit in which $c = 1$, restoring the SI unit can be done with insertion of appropriate fundamental constants under dimensional analysis.

Basically, we first analyze this process in the CM frame, where the mother stay at rest and the two daughters move in opposite directions with the same momentum.

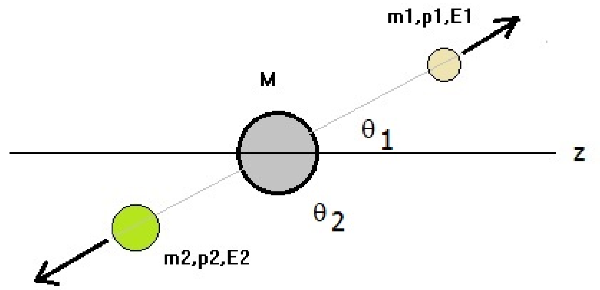

Figure 1:

The kinematics of this decay is determined from the condition of energy-momentum conservation, which is simply written in the form

$$P^\mu = (p_1 + p_2)^\mu \tag{6}$$

In the CM frame, we have

$$P^\mu = (M, \vec{0}),\ p_1^\mu = (E_1, \vec{p}),\ p_2^\mu = (E_1, -\vec{p})$$

Thus we have

$$M = E_1 + E_2 = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} \tag{7}$$

Solving this equation for p, we get

$$p = \frac{1}{2M}\sqrt{[M^2(m_1 - m_2)^2][M^2 - (m_1 + m_2)^2]} \tag{8}$$

This shows us the decay condition that $M \geq m_1 + m_2$.

To complete out calculation for E_1, E_2, we have

$$E_1 = \frac{1}{2M}(M^2 + m_1^2 - m_2^2),\ E_1 = \frac{1}{2M}(M^2 - m_1^2 + m_2^2) \tag{9}$$

For special case of two identical daughters of mass m, we will have $E_1 = E_2 = \frac{1}{2}M$ and $p = \frac{1}{2}\sqrt{M^2 - 4m^2}$

Exercise 5: Let calculate the energy and momentum of the two Pions from the Kaon decay $K^0 \to \pi^+\pi^-$, in the CM frame. (Find the corresponding masses by yourself.)

Back to the real life, this decay occurs in the LAB frame from the energetic mother.

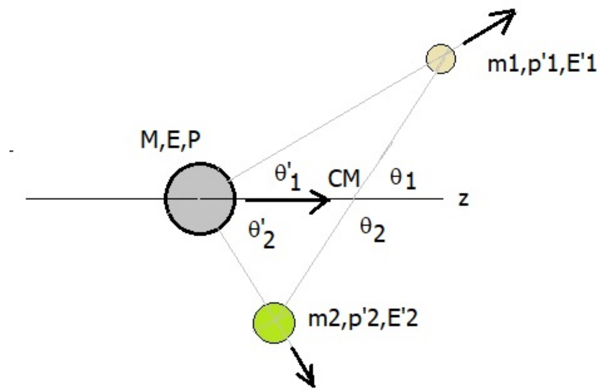

Figure 2:

First we have to write the CM 4-momenta of the daughters as

$$p_1^\mu = (E_1, p_\perp, p_z),\ p_2^\mu = (E_2, -p_\perp, p_z)$$

Next assign the LAB frame 4-momenta

$$P^{*\mu} = (E, 0, 0, P),\ p_1^{*\mu} = (E_1^*, p_{1\perp}^*, p_{1z}^*),\ p_2^{*\mu} = (E_2^*, p_{2\perp}^*, p_{2z}^*)$$

We can have the LT factors as

$$\gamma = \frac{E}{M},\ \beta = \frac{P}{E}\ \ (\text{in z} - \text{direction}) \tag{10}$$

Apply the LT, we have

$$E_1^* = \gamma(E_1 - \beta p_z),\ p_{1z}^* = \gamma(p_z - \beta E_1),\ p_{1\perp}^* = p_\perp \tag{11}$$

Exercise 6: Derive expressions of the LT of daughter particle 2.

Let us determine the relation of emitting angles

$$p_\perp = p\sin\theta_1,\ p_{1\perp}^* = p^*\sin\theta_1',\ p_{1z}^* = p^*\cos\theta_1'$$

$$\tan\theta_1' = \frac{p\sin\theta_1}{\gamma(p\cos\theta_1 - \beta E_1)} = \frac{\sin\theta_1}{\gamma(\cos\theta_1 - \beta v_1)},\quad v_1 = \frac{|\vec{p}|}{E_1} \tag{12}$$

Exercise 6: Determine the decay $K^0 \to \pi^+\pi^-$ with the Kaon energy of $1.0GeV$ in the LAB frame. What is the emitting angles of Pions in the LAB frame, when we determine the 90^0 emitting direction in the CM frame?

Exercise 7: Determine the decay $\Lambda^0 \to p^+\pi^-$, with the Lambda energy of $2.2GeV$ in the LAB frame. Let determine this decay in forward direction.

Exercise 8: Determine the decay $\pi^0 \to \gamma\gamma$, with the Pion energy of 500 MeV, in forward direction.