SCPY322 Nuclear and Particle Physics
Friday 2, April 2021

2 Relativistic Kinematics II

2.1 Reviews of the last lecture

Working in the unit in which $c = 1 = \hbar$.
We have derived the 4-momentum $p^\mu = (E, \vec{p})$, such that

$$p^2 = m^2,\ E = \gamma m,\ \vec{p} = \gamma m \vec{v}$$

when $\gamma = (1 - \beta^2)^{-1/2},\ \beta = v/c$, so that

$$\vec{\beta} = \frac{\vec{p}}{E},\ \gamma = \frac{E}{m}$$

Energy-momentum conservation means $(p_1 + p_2 + ...)^\mu$ = constant, i.e., before and after.
LT (in z-direction) of 4-momentum is

$$E' = \gamma(E - \beta p_z),\ p'_z = \gamma(p_z - \beta E),\ p'_\perp = p_\perp$$
$$E = \gamma(E' + \beta p'_z),\ p_z = \gamma(p_z + \beta E'),\ p_\perp = p'_\perp$$

In particle physics CM-frame moves relative to the rest LAB-frame, relative to an observer.

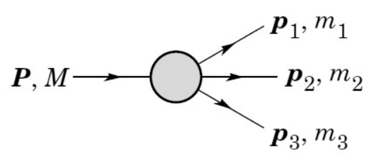

Figure 2.1:

2.2 Three-body decays

Let us determine a decay of a heavy particle of mass M into three particles of masses m_1, m_2, m_3, see figure (2.1) above. The conservation of energy-momentum is

$$P^\mu = (p_1 + p_2 + p_3)^\mu \tag{2.1}$$

Let us define the following Lorentz invariant quantities:

$$\begin{aligned} s &= P^2 = M^2 &(2.2)\\ s_1 &= (P-p_1)^2 = (p_2+p_3)^2 &(2.3)\\ s_2 &= (P-p_2)^2 = (p_1+p_3)^2 &(2.4)\\ s_3 &= (P-p_3)^2 = (p_1+p_2)^2 &(2.5)\end{aligned}$$

Their meanings are $\sqrt{s}$ is the invariant mass of the mother particle, $\sqrt{s_1}$ is the invariant mass of a system of particles 2+3, $\sqrt{s_2}$ is the invariant mass of a system of particles 1+3, and $\sqrt{s_3}$ is invariant mass of a system of particles 1+2. The three invariants s_1, s_2, s_3 are not independent, we can observe from (2.2-2.5) that

$$s_1 + s_2 + s_3 = M^2 + m_1^2 + m_2^2 + m_3^2 \tag{2.6}$$

Exercise 2.1: Derive this expression explicitly.

In the rest frame of mother particle, we will have

$$s_1 = M^2 + m_1^2 - 2ME_1 \tag{2.7}$$

with $E_1 = \sqrt{p_1^2 + m_1^2}$. Since $E_1 \geq m_1$, thus we can observe from (2.7) that

$$s_{1,max} = (M - m_1)^2 \tag{2.8}$$

To find $s_{1,min}$, we have to evaluate s_1 in the rest frame of subsystem (2,3), the Jackson frame [1], which is denote as

$$s_1 = (p_2+p_3)^2 = (E_2^o + E_3^o)^2 \geq (m_2+m_3)^2 = s_{1,min} \tag{2.9}$$

We can similar formula for s_2, s_3. In summary we have the ranges of possible s_1, s_2, s_3 from the decay as

$$\begin{aligned} s_1 &\in [(m_2+m_3)^2, (M-m_1)^2] &(2.10)\\ s_2 &\in [(m_1+m_3)^2, (M-m_2)^2] &(2.11)\\ s_3 &\in [(m_1+m_2)^2, (M-m_3)^2] &(2.12)\end{aligned}$$

In the Jackson frame of a subsystem (2,3), we also observe from energy-momentum conservation that $\vec{p}_1^{\,o} = \vec{P}^o$, thus we have

$$s_1 = (E^o - E_1^o)^2 = \left(\sqrt{M^2 + P^{o2}} - \sqrt{m_1^2 + p_1^{o2}}\right)^2 \tag{2.13}$$

[1] In this frame we denote $p_2^\mu = (E_2^o, \vec{p}_2^{\,o})$ and $p_3^\mu = (E_3^o, \vec{p}_3^{\,o})$, in which $\vec{p}_3^{\,o} = -\vec{p}_2^{\,o}$.

Solving for p_1^{o2}, we have

$$p_1^{o2} = \frac{1}{4s_1}[s_1 - (M - m_1)^2][s_1 + (M + m_1)^2] = \frac{1}{4s_1}\lambda(s_1, M^2, m_1^2) \quad (2.14)$$

where

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \quad (2.15)$$

is known as *Stueckelberg* kinematics function. The corresponding expressions for p_2^{o2}, p_3^{o2} are

$$p_2^{o2} = p_3^{o2} = \frac{1}{4s_1}\lambda(s_1, m_2^2, m_3^2) \quad (2.16)$$

Exercise 2.2: Derive details expressions of p_2^{o2} and p_3^{o2}.

Next let us determine the invariant s_2, in the Jackson frame of subsystem (2,3) we observe that

$$s_2 = (p_1 + p_3)^2 = m_1^2 + m_3^2 + 2(E_1^o E_3^o - p_1^o p_3^o \cos\alpha) \quad (2.17)$$

If s_1 is fixed, we can see that s_2 depends only on α and shows up its maximum $s_{2+} = s_{2,max}$ and minimum $s_{2-} = s_{2,min}$ values as

$$s_{2\pm} = m_1^2 + m_3^2 + 2(E_1^o E_3^o \pm p_1^o p_3^o) \quad (2.18)$$

Rewrite E_1^o, E_3^o in terms of s_1, as

$$E_1^o = \frac{1}{2\sqrt{s_1}}(s - s_1 - m_1^2), \; E_2^o = \frac{1}{2\sqrt{s_1}}(s_1 - m_2^2 + m_3^2) \quad (2.19)$$

Then we have from (2.18)

$$s_{2\pm} = m_1^2 + m_3^2 + \frac{1}{2s_1}\left[(s - s_1 + m_1^2)(s_1 - m_2^2 + m_3^2) \right.$$
$$\left. \pm\lambda^{1/2}(s, s_1, m_1^2)\lambda^{1/2}(s_1, m_2^2, m_3^2)\right] \quad (2.20)$$

The curve defined by (2.20) is defined the boundary of *the Dalitz plot* on the (s_1, s_2) plane, see figure (2.1). See an example of Dalitz plot of $D^+ \to K^- D^+ D^+$ decay, at *https://arxiv.org/pdf/1902.05884.pdf*.

Exercise 2.3: Determine the Dalitz plot of $K^+ \to \pi^0 \mu^+ \nu_\mu$ decay.

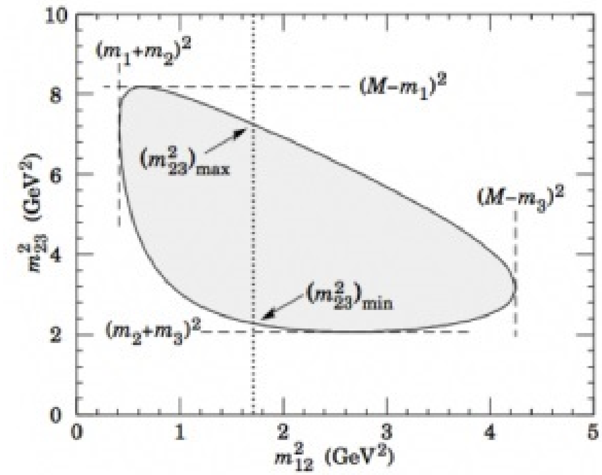

Figure 2.2:

2.3 Decay rate

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2.4 Two-particle collisions

Let us determine two-to-two particle collision $a + b \to c + d$, see figure (2.3).

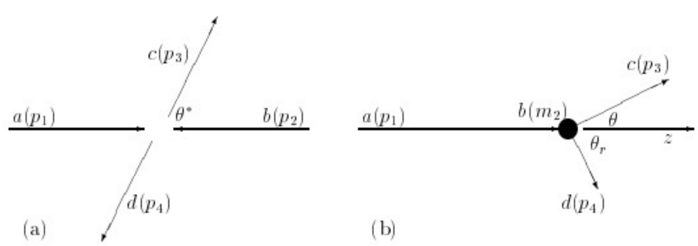

Figure 2.3:

The energy-momentum conservation reads

$$(p_1 + p_2)^\mu = (p_3 + p_4)^\mu \tag{2.21}$$

In the LAB frame, we have

$$p_1^\mu = (E_1, 0, 0, p_1),\ \ p_2^\mu = (m_2, 0, 0, 0)$$

$$p_3^\mu = (E_3, 0, p_3 \sin\theta, p_3 \cos\theta),\ \ p_4^\mu = (E_4, 0, -p_4 \sin\theta_r, p_4 \cos\theta_r)$$

In the MC frame we have

$$p'^\mu_1 = (E'_1, 0, 0, p'_1),\ \ p'^\mu_2 = (E'_2, 0, 0, -p'_2),\ \ p'_2 = p'_1 \tag{2.22}$$

$$p'^\mu_3 = (E'_3, 0, p'_3 \sin\theta', p'_3 \cos\theta'),\ \ p'^\mu_4 = (E'_4, 0, -p'_4 \sin\theta', -p'_4 \cos\theta') \tag{2.23}$$

with $\theta'_r = \pi - \theta'$ and $p'_4 = p'_3$. The Lorentz invariant quantity s is

$$s = (p'_1 + p'_2)^2 = (E'_1 + E'_2)^2 \tag{2.24}$$

$$E'_1 = \sqrt{p'^2_1 + m_1^2},\ \ E'_2 = \sqrt{p'^2_1 + m_2^2} \tag{2.25}$$

$$\to p'_1 = \frac{1}{2\sqrt{s}} \sqrt{[s - (m_1 - m_2)^2][s - (m_1 + m_2)^2]} \tag{2.26}$$

From energy-momentum conservation, we also have

$$s = (p'_3 + p'_4)^2 = (E'_3 + E'_4)^2 \tag{2.27}$$

$$E'_3 = \sqrt{p'^2_3 + m_3^2},\ \ E'_4 = \sqrt{p'^2_3 + m_4^2} \tag{2.28}$$

$$p'_3 = \frac{1}{2\sqrt{s}} \sqrt{[s - (m_3 - m_4)^2][s - (m_3 + m_4)^2]} \tag{2.29}$$

Back to the LAB frame, we have

$$s = (p_1 + p_2)^2 = m_1^2 + m_2^2 = 2m_2 E_1 \tag{2.30}$$

$$\to E_1 = \frac{s - m_1^2 - m_2^2}{2m_2} \equiv \sqrt{p_1^2 + m_1^2} \tag{2.31}$$

$$\begin{aligned} \to p_1 &= \frac{1}{2m_2} \sqrt{[s - (m_1 - m_2)^2][s - (m_1 + m_2)^2]} \\ &= \frac{1}{2m_2} \lambda^{1/2}(s, m_1^2, m_2^2) \end{aligned} \tag{2.32}$$

From (2.26) and (2.32), we observe that

$$p'_1 = p_1 \frac{m_2}{\sqrt{s}} \to E'_{1,2} = \frac{m_{1,2}^2 + m_2 E_1}{\sqrt{s}} \tag{2.33}$$

To derive the LT parameters, let us determine the total 4-momentum $p^\mu = (p_1 + p_2)^\mu$, its LT in z-direction is

$$E_1 + m_2 = \gamma_{cm}((\underbrace{E_1' + E_2'}_{\sqrt{s}}) + v_{cm}(\underbrace{p_1' + p_2'}_{=0})) = \gamma_{cm}\sqrt{s} \tag{2.34}$$

$$\gamma_{cm} = \frac{E_1 + m_2}{\sqrt{s}}, \quad \text{and } v_{cm} = \frac{p_1}{E_1 + m_2} \tag{2.35}$$

Let us test by calculating the LT of $E_{1,2} \to E'_{1,2}$, we have

$$E_1' = \gamma_{cm}(E_1 - v_{cm}p_1) = \frac{m_1^2 + m_2 E_1}{\sqrt{s}} \tag{2.36}$$

$$E_2' = \gamma_{cm}(m_2 - v_{cm}0) = \frac{m_2(m_2 + E_1)}{\sqrt{s}} \tag{2.37}$$

We derive (2.33).

2.4.1 Elastic collision

We have $p_1 + p_2 = p_3 + p_4$, where

$$m_3^2 = m_1^2, \; p_4^2 = m_2^2 \tag{2.38}$$

We define the Lorentz invariant quantities

$$s = (p_1 + p_2) = (p_3 + p_4)^2 \tag{2.39}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \tag{2.40}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \tag{2.41}$$

$$\to s + t + u = 2m_1^2 + 2m_2^2 \tag{2.42}$$

In the CM frame, we have, with $E_1' = E_2'$,

$$t = -(\vec{p}_1' - \vec{p}_3')^2 = -2p_1'^2(1 - \cos\theta') \tag{2.43}$$

This shows that t is related to scattering angle, and in the LAB frame, we have

$$t = (p_2 - p_4)^2 = 2m_2(m_2 - E_4) = -2m_2 T_4 \tag{2.44}$$

where $T_4 = E_4 - m_2$ the recoil kinetic energy.

2.4.2 Inelastic collision

We have $p_1 + p_2 = p_3 + p_4 + ...$ In the LAB frame we have

$$p_1^\mu = (E_1, \vec{p}), \;\; p_2^\mu = (m_2, 0) \tag{2.45}$$

$$s = (p_1 + p_2)^2 = m_1^2 + m_2 + 2m_2 E_1 \tag{2.46}$$

In the CM frame, we also have

$$s = (p_3' + p_4' + ...)^2 \geq (m + 3^2 + m + 4^2 + ...), \quad E_i' = \sqrt{m_i^2 + p_i'^2} \tag{2.47}$$

$$E_{thr}' = \sqrt{s_{min}} = m_3 + m_4 + ... \equiv M \tag{2.48}$$

$$\rightarrow E_{1,thr} = \frac{1}{2m_2}[M^2 - m_1^2 - m_2^2] \tag{2.49}$$

$$T_1^{thr} = E_{1,thr} - m_1 = \frac{1}{2m_2}[M^2 - (m_1 + m_2)^2] \tag{2.50}$$

For example of the inelastic scattering $\pi^+ + p \rightarrow \pi^+ + p + \pi^+ + \pi^-$, we have $M(p) = 940MeV, M(\pi) = 140MeV$, so that

$$T_\pi^{thr} = 363.4MeV$$

Exercise 2.4 Evaluate T_π^{th} above explicitly.

2.5 Cross section

The differential cross section is defined to be

$$d\sigma = \frac{\text{scattering rate } W_{fi} \text{ into solid angle } d\Omega}{\text{Incident flux } F} \tag{2.51}$$

$$\frac{d\sigma}{d\Omega} = \frac{W_{fi}}{F} \tag{2.52}$$