7 Interaction Perturbation and Diagrammatic

7.1 Interacting Hamiltonian

Interacting Hamiltonian

$$H = H_0 + V, \quad H_0 = \int d^3x \psi^\dagger(x) \frac{p^2}{2m} \psi(x) \tag{7.1}$$

$$V = \frac{1}{2} \int d^3x \int d^3x' \psi^\dagger(x)\psi^\dagger(x') V(x - x')\psi(x')\psi(x) \tag{7.2}$$

7.2 Interacting Green's function

Define as

$$iG(x, x'; t) = \frac{\langle\Omega|T[\psi(x, t)\psi^\dagger(x, 0)]|\Omega\rangle}{\langle\Omega|\Omega\rangle} \tag{7.3}$$

where $|\Omega\rangle$ is interacting ground state and $\langle\Omega|\Omega\rangle \neq 1$. Since $\psi(x, t) = \psi_H(x, t)$, rewrite in term of interacting operator as

$$\psi_H(x, t) = U_I^\dagger(t, 0)\psi_I(x, t)U_I(t, 0)$$

From above we will have

$$iG(x, x'; t) = \frac{\langle\Omega|T[U_I^\dagger(t, 0)\psi_I(x, t)U_I(t, 0)\psi_I^\dagger(x, 0)]|\Omega\rangle}{\langle\Omega|\Omega\rangle} \tag{7.4}$$

7.3 Gell-Mann and Low theorem

The theorem state that *the interaction is adiabtaically turned on from the far past and turned off in the far future. It has a full strength of interaction at the present.* The system Hamiltonian can be modified into the form

$$H = H_0 + e^{-\eta|t|}V, \ \eta \to 0 \tag{7.5}$$

We can connect the interacting ground state to the non-interacting one in the form

$$|\Omega\rangle = \lim_{T\to\infty} U_I(0, -T)|\Omega_0\rangle \tag{7.6}$$

$$\langle\Omega| = \lim_{T\to\infty} \langle\Omega_0|U_I^\dagger(0, -T) = \lim_{T\to\infty} \langle\Omega_o|U_I(T, 0) \tag{7.7}$$

Insertion into (7.4), we have

$$
\begin{aligned}
iG(x,x';t) &= \lim_{T\to\infty} \frac{\langle\Omega_0|T[U_I(T,t)\psi_I(x,t)U_I(t,0)\psi_I^\dagger(x',0)U_I(0,-T)]|\Omega_0\rangle}{\langle\Omega_0|U_I(T,-T)|\omega_0\rangle} \\
&= \frac{\Omega_0|T[U_I(\infty,-\infty)\psi_I(x,t)\psi_I^\dagger(x',0)]|\Omega_0\rangle}{\langle\Omega_0|U_I(\infty,-\infty)|\Omega_0\rangle} \equiv \frac{N}{D} \qquad (7.8)
\end{aligned}
$$

7.4 Perturbation theory

According to the fact that

$$U_I(\infty,-\infty) = T\exp\left\{-i\int_{-\infty}^{\infty} dt' V_I(t')\right\} \qquad (7.9)$$

From (7.8) one can write

$$
\begin{aligned}
D &= \sum_{n=0}^{\infty} D^{(n)} &(7.10)\\
D^{(0)} &= 1 &(7.11)\\
D^{(1)} &= -i\int dt_1 \langle\Omega_0|T[V_I(t_1)]|\Omega_0\rangle &(7.12)\\
D^{(2)} &= \frac{(-i)^2}{2}\int dt_1 \int dt_2 \langle\Omega_0|T[V_I(t_1)V_I(t_2)]|\Omega_0\rangle &(7.13)\\
... &\quad ... \\
D^{(n)} &= \frac{(-i)^n}{n!}\int dt_1 ... \int dt_n \langle\Omega_0|T[V_I(t_1)...V_I(t_n)]|\Omega_0\rangle &(7.14)
\end{aligned}
$$

and

$$
\begin{aligned}
N &= \sum_{n=0}^{\infty} N^{(n)} &(7.15)\\
N^{(0)} &= \langle\Omega_0|T[\psi_I(x,t)\psi_I^\dagger(x',0)]|\Omega_0\rangle = iG(x,x';t) &(7.16)\\
N^{(1)} &= -i\int dt_1 \langle\Omega_0|T[V_I(t_1)\psi_I(x,t)\psi_I^\dagger(x',0)]|\Omega_0\rangle &(7.17)\\
N^{(2)} &= \frac{(-i)^2}{2}\int dt_1 \int dt_2 \langle\Omega_0|T[V_I(t_1)V_I(t_2)\psi_I(x,t)\psi_I^\dagger(x',0)]|\Omega_0\rangle &(7.18)\\
... &\quad ... \\
N^{(n)} &= \frac{(-i)^n}{n!}\int dt_1 ... \int dt_2 \langle\Omega_0|T[V_I(t_1)...V_I(t_n)\psi_I(x,t)\psi_I^\dagger(x',0)]|\Omega_0\rangle &(7.19)
\end{aligned}
$$

7.5 Wick's theorem

The theorem state that *product of number of operators can be written in term of the summation of* **normal ordering** *and all possible* **contractions**.

$$ABCD =: ABCD : +[AB][CD] \pm [AC][BD] + [AD][BC]$$

The normal ordering means all annihilation operators are on the right of creation operators, so that

$$\langle\Omega_0| : ABCD : |\Omega_0\rangle = 0$$

The contraction means a pair of time-ordered operators

$$\langle\Omega_0|[AB]|\Omega_0\rangle = iG(A, B)$$

From (7.12), we observe that

$$\begin{aligned} D^{(1)} &= \frac{-i}{2}\int dt_1 \int d^3x_1 \int d^3x_1' V(x_1 - x_1') \\ &\quad \times\langle\Omega_0|T[\psi_I^\dagger(x_1, t_1)\psi_I^\dagger(x_1', t_1)\psi_I(x_1', t_1)\psi_I(x_1, t_1)|\Omega_0\rangle \\ &= \frac{-i}{2}\int dt_1 \int d^3x_1 \int d^3x_1' V(x_1 - x_1') \\ &\quad \big\{iG(x_1, x_1; 0)iG(x_1', x_1'; 0) - iG(x_1, x_1'; 0)iG(x_1', x_1; 0)\big\} \end{aligned} \tag{7.20}$$

From (7.17), we have

$$\begin{aligned} N^{(1)} &= \frac{-i}{2}\int dt_1 \int d^3x_1 \int d^3x_1' V(x_1 - x_1') \\ &\quad \times\langle\Omega_0|T[\psi_I^\dagger(x_1, t_1)\psi_I^\dagger(x', t_1)\psi_I(x_1', t_1)\psi(x_1, t_1)\psi_I(x, t)\psi_I^\dagger(x', 0)]|\Omega_0\rangle \\ &= \frac{-i}{2}\int dt_1 \int d^3x_1 \int d^3x_1' V(x_1 - x_1') \\ &\quad \big\{iG(x, x'; t)\left[iG(x_1, x_1; 0)iG(x_1', x_1'; 0) - iG(x_1, x_1'; 0)iG(x_1', x_1; 0)\right] \\ &\qquad +iG(x, x_1; t - t_1)iG(x_1', x_1'; 0)iG(x_1, x'; t_1) \\ &\qquad -iG(x, x_1; t - t_1)iG(x_1, x_1'; 0)iG(x_1', x'; t_1)\big\} \end{aligned} \tag{7.21}$$

7.6 Diagrammatic

Let us assign the diagrammatic rules

- Particle Green's function $iG(x, x'; t)$, see Figure (7.1a)

- Interaction potential $V(x - x')$, see Figure (7.1b)
- Vertex $\frac{-i}{2}$, see Figure (7.1c)
- Integrate overall space and time points
- Insert a factor of $1/n!$ for the n^{th}-order diagram

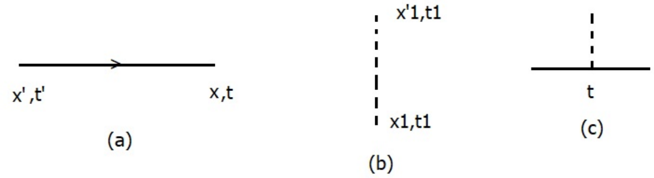

Figure 7.1:

Note that the time direction is from left to right. From (7.20,21), see Figure (7.2)

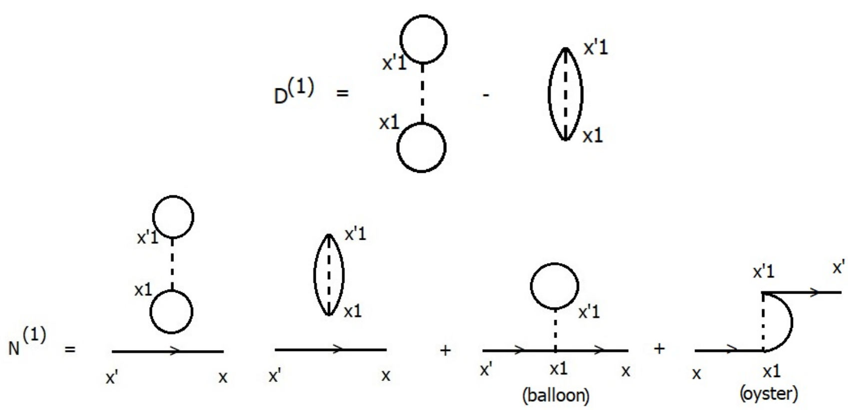

Figure 7.2:

7.7 Linked cluster theorem

In general the D-term contain all disconnected diagrams, while the N-term can be factorized to be the product of connected and disconnected diagrams. This results to have only connected diagram from the N-term for the Green's function. This is called *linked cluster theorem*, see Figure (7.3)

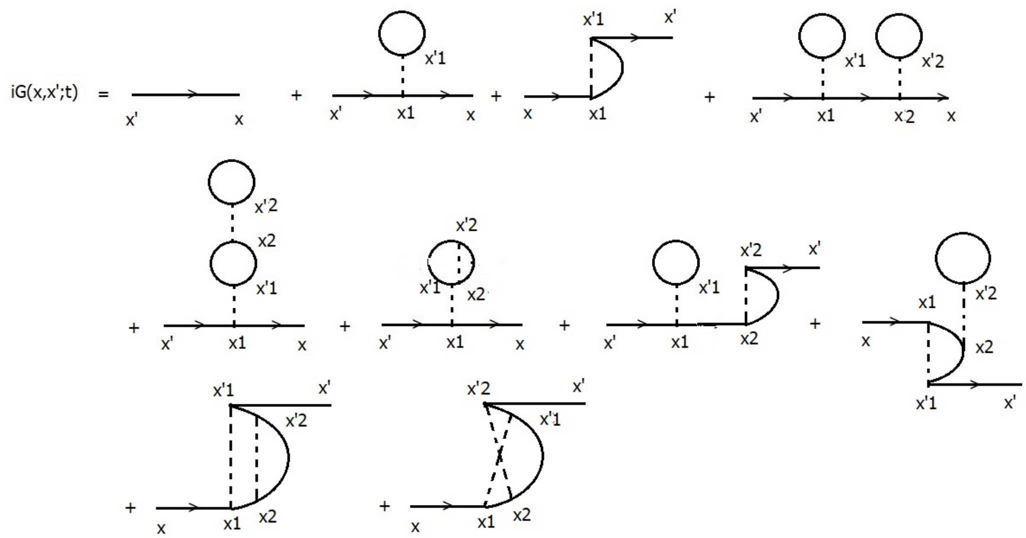

Figure 7.3:

Note that bare Green's function (zero-order) diagram has no interaction line, first-order diagrams have one interaction line, second-order diagrams have two interaction lines, and so on.

7.8 Diagram on momentum space

Apply with the Fourier transformation

$$iG(x, x'; t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{ik\cdot(x-x')-i\omega t} iG(k, \omega) \tag{7.22}$$

$$iG(k, \omega) = \int d^3x \int dt e^{-ik\cdot(x-x')+i\omega t} iG(x, x'; t) \tag{7.23}$$

$$V(x - x') = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot(x-x')} V(q) \tag{7.24}$$

$$V(q) = \int d^3x e^{-iq\cdot(x-x')} V(x - x') \tag{7.25}$$

We already know an expression of the Green's function

$$iG(k, \omega) = \frac{1}{\omega - \xi_k \pm i\eta}, \ \xi_k = \epsilon_k - \mu \tag{7.26}$$

and $\mu = E_F$. From Figure (7.4), we have

$$\begin{aligned} iG_H(x,x';t) = \frac{(-i)}{2}\int dt_1 \int d^3x_1 \int d^3x'_1 V(x_1 - x'_1) iG(x,x_1;t-t_1) \\ \ldots \times iG(x'_1,x'_1;0)iG(x'_1,x';t_1) \\ = \frac{-i}{2}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\int \frac{d^3k''}{(2\pi)^3}\int \frac{d\omega''}{2\pi} \\ \ldots \times V(q)iG(k,\omega)iG(0,0)iG(k'',\omega'') \\ \ldots \times \int dt_1 \int d^3x_1 \int d^3x'_1 e^{ik\cdot x - ik''\cdot x' + i(q-k+k'')\cdot x_1 - iq\cdot x'_1 - i\omega t - i(\omega''-\omega)t_1} \end{aligned} \tag{7.27}$$

Space and time integration results with

$$2\pi\delta(\omega'' - \omega)(2\pi)^3\delta(q)(2\pi)^3\delta(q - k + k'')$$

Then we have

$$\begin{aligned} iG_H(x,x';t) &= \int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi} e^{ik\cdot(x-x') - i\omega t} \\ &\times \frac{-i}{2} V(0)iG(k,\omega)iG(0,0)iG(k,\omega) \end{aligned} \tag{7.28}$$

$$\rightarrow iG_H(k,\omega) = \frac{-i}{2} V(0)iG(k,\omega)iG(0,0)iG(k,\omega) \tag{7.29}$$

See the diagram in Figure (7.4a).

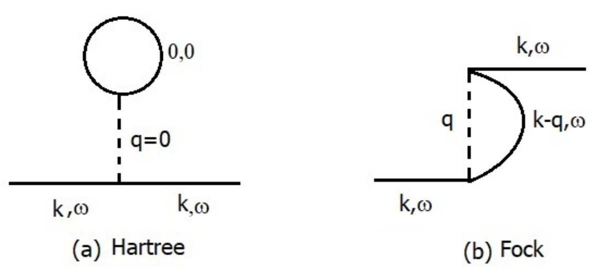

Figure 7.4:

A similar expression for the Fock term, see Figure (7.4b), will be

$$iG_F(k,\omega) = \frac{i}{2}\int \frac{d^3k'}{(2\pi)^3} V(k-k')iG(k,\omega)iG(k-k',\omega)iG(k,\omega) \tag{7.30}$$

Diagrammatic rules on momentum space:

- Particle Green's function $iG(k,\omega)$, see Figure (7.5a)
- Interaction potential $V(q)$, see Figure (7.5b)
- At each interaction vertex apply with a factor $\Gamma = \sqrt{\frac{-i}{2}}$, see Figure (7.5c)
- Apply conservation of energy-momentum (ω, k) at each vertex
- Apply loop momentum integration $\int \frac{d^3k}{(2\pi)^3}$, also apply the loop energy integration $\int \frac{d\omega}{2\pi}$ for the time-dependent interaction

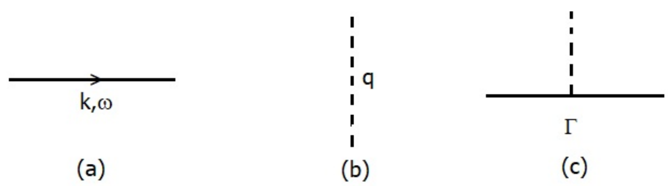

Figure 7.5:

7.9 Dyson's equation

From above, we can observe that the interacting Green's function can be written in the form

$$iG = iG_0 + iG_0(...)iG_0 \tag{7.31}$$

Let us define $\Sigma(k,\omega) = \Sigma^H(k,\omega) + \Sigma^F(k,\omega)$ as the *particle self-energy*, the Green's function can be expressed in the form

$$\begin{aligned} iG &= iG_0 + iG_0\Sigma iG_0 + iG_0\Sigma iG_0\Sigma iG_0 + ... \\ &= iG_0 + iG_0\Sigma(iG_0 + iG_0\Sigma iG_0 + iG_0\Sigma iG_0\Sigma iG_0 + ...) \qquad (7.32) \\ &= iG_0 + iG_0\Sigma iG \qquad (7.33) \end{aligned}$$

This is called *Dyson's equation*, it can be solved in the form

$$(1 - iG_0\Sigma)iG = iG_0((iG_0)^{-1} - \Sigma)iG_0 = iG_0$$
$$(iG)^{-1} = (iG_0)^{-1} - \Sigma \to iG(k,\omega) = \frac{1}{\omega - \xi_k - \Sigma(k,\omega)} \tag{7.34}$$

This show that particle self energy make a shift in particle spectrum, i.e., $\xi_k \to \xi_k + \Sigma(k,\omega)$. The particle self-energy will be determined from a particular kind of particle interaction, anyway its generic diagrammatic representation appear in Figure (7.6).

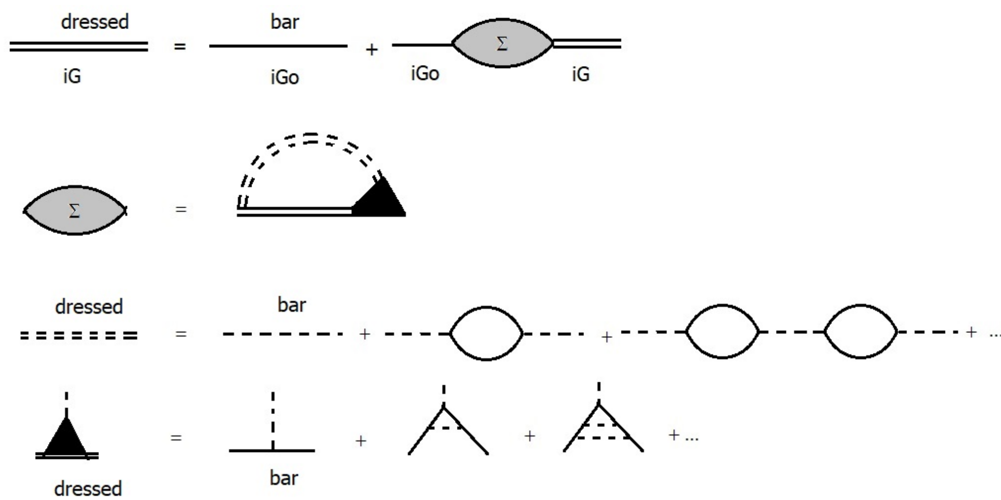

Figure 7.6:

Note that these relations are self-consistent. Note also the we have defined *bar* and *dressed* quantities.

7.10 Random phase approximation: RPA

At the first order correction for both for Green's function and self energy, contain Hartree term (balloon diagram) and Fock term (oyster diagram). At the second order we have RPA approximation, it contain the bubble grams, see Figure (7.7), and it is written as

$$\Sigma_{RPA} = \sum_{n=1}^{\infty} \Sigma_{RPA}^{(n)} \tag{7.35}$$

In the RPA diagrams, they contain bar Green's function, bar vertex, and bar interaction.

Let us define *medium polarization* $\Pi(q)$, see Figure (7.8). One can write

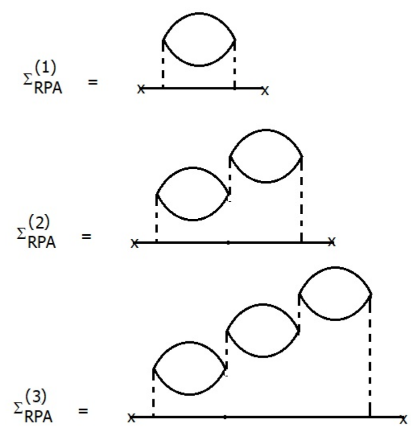

Figure 7.7:

dresses interaction potential in the form

$$\tilde{V}(q) = V(q) + v(q)\Pi(q)V(q) + V(q)\Pi(q)V(q)\Pi(q)V(q) + ... \\ = V(q) + V(q)\Pi(q)\tilde{V}(q) \quad (7.36)$$

$$\tilde{V}(q) - V(q)\Pi(q)\tilde{V}(q) = V(q) \to \tilde{V}(q) = \frac{V(q)}{1 - V(q)\Pi(q)} \equiv \frac{V(q)}{\epsilon(q)} \quad (7.37)$$

We have defined dielectric function

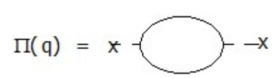

Figure 7.8:

$$\epsilon(q) = 1 - V(q)\Pi(q) \quad (7.38)$$

In case of Coulomb interaction, $\tilde{V}(q)$ will be screened Coulomb interaction.