13 Gravity Beyond Einstein's Theory

Einstein's equation of gravity, with matter coupling, is derived from least action principle of the Einstein-Hilbert action

$$S[g] = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}R + \mathcal{L}_{matter}\right]$$

with $\kappa = 8\pi G$ and $g = \det[g_{\mu\nu}]$. The extension of Einstein's theory will start from this action.

13.1 Extended theory of gravity

13.1.1 Brans-Dicke theory

The Brans-Dicke theory of gravity is the prototype gravitational theory alternative to Einstein's theory. The action in *Jordan frame* $(g_{\mu\nu}, \phi)$, where $\phi(x)$ is the scalar field, is

$$S[g,\phi] = \frac{1}{16\pi}\int d^4\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \mathcal{V}(\phi)\right] + \int d^4x\sqrt{-g}\mathcal{L}_{matter} \tag{13.1}$$

From least action principle, the variation with respect to $g_{\mu\nu}$ will give us the following terms

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{13.2}$$

$$\delta\left(\sqrt{-g}R\right) = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g_{\mu\nu} = G_{\mu\nu}\delta g_{\mu\nu} \tag{13.3}$$

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{matter}\right)}{\delta g^{\mu\nu}}\delta g^{\mu\nu} \tag{13.4}$$

The field equation becomes

$$G_{\mu\nu} = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla\phi\cdot\nabla\phi\right) + \frac{1}{\phi}\left(\nabla_\mu\phi\lambda_\nu\phi - g_{\mu\nu}\nabla^2\phi\right) - \frac{1}{2\phi}\mathcal{V}g_{\mu\nu} \tag{13.5}$$

The variation with respect to ϕ gives us the field equation

$$\frac{2\omega}{\phi}\nabla^2\phi + R - \frac{\omega}{\phi}\nabla\phi\cdot\nabla\phi - \frac{d\mathcal{V}}{d\phi} = 0 \tag{13.6}$$

Taking the trace of (13.5), with $G = -R$, we have

$$R = \frac{-8\pi T}{\phi} + \frac{\omega}{\phi^2}\nabla\phi\cdot\nabla\phi + \frac{3\nabla^2\phi}{\phi} + \frac{2\mathcal{V}}{\phi} \tag{13.7}$$

where $T = g^{\mu\nu}T_{\mu\nu}$. After insertion into (13.6), we have

$$\nabla^2\phi = \frac{1}{2\omega+3}\left(8\pi T + \phi\frac{d\mathcal{V}}{d\phi} - 2\mathcal{V}\right) \tag{13.8}$$

For Klein-Gordon scalar field ϕ, the field equation (13.5)will appear as

$$G(\phi) \propto \frac{1}{\phi} \tag{13.9}$$

This is known as *Mach's principle.*

13.1.2 $f(R)$-theory

Let us determine gravity from a simple $f(R) = R + \alpha R^2$ theory. The action becomes

$$S[g] = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(R + \alpha R^2 + 2\kappa\mathcal{L}_{matter}\right) \tag{13.10}$$

With the variation

$$\delta\left(\sqrt{-g}R^2\right) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}R^2\delta g^{\mu\nu} + 2\sqrt{-g}R\delta R \tag{13.11}$$

$$\mapsto R\delta R = R\left(\delta(g^{\mu\nu})\delta R_{\mu\nu} + g^{\mu\nu}\delta(R_{\mu\nu})\right) \tag{13.12}$$

$$\text{and } g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu\nabla_\nu h^{\mu\nu} - \nabla^2 h \tag{13.13}$$

where $h^{\mu\nu} = -\delta g^{\mu\nu}$ and $h = -g_{\mu\nu}\delta g^{\mu\nu}$. Then one have (13.13)

$$\int d^4x\sqrt{-g}Rg^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\sqrt{-g}R\left(\nabla_\mu\nabla_\nu h^{\mu\nu} - \nabla^2 h\right) \tag{13.14}$$

$$\xrightarrow[2\ times]{integration\ by\ parts} \int d^4x\sqrt{-g}\left(h^{\mu\nu}\nabla_\mu\nabla_\nu R - h\nabla^2 R\right)$$

$$= -\int d^4x\sqrt{-g}\left(\nabla_\mu\nabla_\nu R - g_{\mu\nu}\nabla^2 R\right)\delta g^{\mu\nu} \tag{13.15}$$

And (13.11) will take the form

$$\begin{aligned}\delta\left(\sqrt{-g}R^2\right) &= \sqrt{-g}\left(2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2\right)\delta g^{\mu\nu} \\ &\quad +2\sqrt{-g}\left(g_{\mu\nu}\nabla^2 R - \nabla_\mu\nabla_\nu R\right)\delta g^{\mu\nu}\end{aligned} \tag{13.16}$$

The gravitational field equation will appear in the form

$$G_{\mu\nu} + \alpha \left[2R \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) + 2 \left(g_{\mu\nu} \nabla^2 R - \nabla_\mu \nabla_\nu R \right) \right] = \kappa T_{\mu\nu} \quad (13.17)$$

Taking the trace, we will have

$$\nabla^2 R - \frac{1}{6\alpha} (R + \kappa T) = 0 \quad (13.18)$$

Let us define

$$\frac{1}{6\alpha} = \omega^2 = m^2 \quad (13.19)$$

$$\xrightarrow{(13.18)} \nabla^2 R - m^2 (R + \kappa T) = 0 \quad (13.20)$$

Is appears as *effective Klein-Gordon equation* of scalar function R, which sometimes is called *scalaron*.

13.2 Alternative formulations

13.2.1 Vierbien field theory of gravity

Reference: Jeffrey Yepez, *Einstein's vierbien field theory of curved space*, arXiv: gr-qc/1106.2037

From the local metric tensor $g_{\mu\nu}(x)$ of the curved space-time manifold M^4, we can observe the vierbien field in the form

$$g_{\mu\nu}(x) = e_\mu(x) e_\nu(x), \; g^{\mu\nu}(x) = e^\mu(x) e^\nu(x) \quad (13.21)$$

$$\mapsto A \in M^4 : A = A^\mu e_\mu, \; A = A_\mu e^\mu \quad (13.22)$$

$$e_\mu = \partial_\mu, \; e^\mu = dx^\mu \mapsto e_\mu e^\nu = \delta_\mu^\nu \quad (13.23)$$

It connects to the *local basis* $(\epsilon_a, \; \epsilon^a)$ in the form

$$\eta_{ab} = \epsilon_a \epsilon_b, \eta^{ab} = \epsilon^a \epsilon^b \mapsto \epsilon_a \epsilon^b = \delta_a^b \quad (13.24)$$

$$e_\mu(x) = e_\mu^a(x) \epsilon_a, \; e^\mu(x) = e_a^\mu(x) \epsilon^a \mapsto e_\mu^a e_b^\mu = \delta_b^a, \; e_\mu^a e_a^\nu = \delta_\mu^\nu \quad (13.25)$$

Note that η_{ab} is Minkowski metric tensor with Lorentzian indices $a, b, ... = 0, 1, 2, 3,$ and $g_{\mu\nu}(x)$ is Riemann metric tensor with Riemannina indices $\mu, \nu, ... = 0, 1, 2, 3.$ And $e_\mu^a(x)$ is known as *vierbien field*, it connects M^4 to TM^4 (tangent of M) at x.

13.2.2 Teleparallel gravity as a gauge theory

Reference: Sebastian Bahamonde et. al., *Teleparallel Gravity-From Theory to Cosmology*, arXiv: gr-qc/2106.13739