10 Kaluza-Klein Theory and Extra Dimensions

From Einstein's equation og gravity, its solution is local metric tensor of curved manifold which is equivalent to the existence gravity from massive object. Actually we have many guest metric tensor for particular gravity or black hole. The metric tensor was also constructed in 4+1 dimension for unification of gravity and electromagnetism done by Kaluza and Klein. And later this idea is extended to dimension greater than 5.

10.1 Kaluza-Klein theory

For simplicity we will work with unit in which $\hbar = c = G1$. Let $\mu, \nu, ... = 0, 1, 2, 3$ are tensor indices of M^4 and $A, B, C, ... = 0, 1, 2, 3, 4$ are tensor indices of M^5. (M^d is d-dimensional curved manifold.) The Kaluza-Klein (KK) metric tensor is constructed on M^5 in the form

$$\hat{g}_{AB}(x) = \begin{pmatrix} g_{\mu\nu}(x) + \phi^2(x)A_\mu(x)A_\nu(x) & \phi^2(x)A_\mu(x) \\ \phi^2(x)A_\nu & \phi^2(x) \end{pmatrix} \tag{10.1}$$

when $\phi(x)$ is a scalar field and $A_\mu(x)$ is a vector field defined on M^4. We can observe from above that

$$\hat{G}^{AB} = \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & g_{\alpha\beta}A^\alpha A^\beta - \frac{1}{\phi^2} \end{pmatrix} \tag{10.2}$$

$$d\hat{s}^2 = \hat{g}_{AB} d\hat{x}^A d\hat{x}^B = g_{\mu\nu}dx^\mu dx^\nu + 2A_\mu dx^\mu dy + \phi^2 dydy \tag{10.3}$$

From the geometrical point of view of the corresponding M^5 we will have

$$\hat{\Gamma}^C_{AB} = \frac{1}{2}\hat{g}^{CD}\left(\partial_A \hat{g}_{DB} + \partial_B \hat{g}_{DA} - \partial_D \hat{g}_{AB}\right) \tag{10.4}$$

$$\hat{R}_{AB} = \partial_C \hat{\Gamma}^C_{AB} - \partial_B \hat{\Gamma}^C AC + \hat{\Gamma}^C_{AB}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\hat{\Gamma}^D_{BC} \tag{10.5}$$

$$\hat{G}_{AB} = \hat{R}_{AB} - \frac{1}{2}\hat{g}_{AB}\hat{;} \;\; \hat{R} = \hat{g}^{AB}\hat{R}_{AB} \tag{10.6}$$

From dynamical point of view, we will have

$$\hat{S} = \frac{1}{16\pi}\int d^4x \int dy \sqrt{-\hat{g}}\hat{R} \tag{10.7}$$

$$\mapsto \delta S = 0 \rightarrow \hat{G}_{AB} = 0 \leftrightarrow \hat{R}_{AB} = 0 \tag{10.8}$$

From (14.1), applying with *cylindrical condition*, i.e. $M^5 = M^4 \times S^1$, we we will have (14.5) in the form

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g}\phi(x) \left(R + \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} + \frac{2\partial^2\phi}{3\phi} \right) \tag{10.9}$$

And $\delta S = 0$ leads to

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \frac{1}{\phi}[\partial_\mu\partial_\nu\phi - g_{\mu\nu}\partial^2\phi] \tag{10.10}$$

$$\partial^\mu F_{\mu\nu} = -\frac{3}{\phi} F_{\mu\nu}\partial^\mu\phi \tag{10.11}$$

$$\partial^2\phi = 4\pi G\phi^3 F_{\mu\nu}F^{\mu\nu} \tag{10.12}$$

with $T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} - g^{\alpha\beta}F_{\mu\alpha}F_{\beta\nu}$ is the Maxwell energy-momentum tensor. Note that when $\phi = 1$ we get a unified description of non-coupling Einstein gravity and Maxwell electromagnetism, as proposed by Kaluza and Klein.

10.2 KK compactifications

Now let us study dimensional reduction or compactification of fields in higher dimensional manifold into product of the lower ones, i.e., in KK theory the reduction is $M^5 \to M^4 \times S^1$.

10.2.1 Compactification of scalar field

For a real scalar field $\phi(x, y)$ which is defined on M^5, with $x \in M^4$ and $y \in S^1$, the compactification is done by doing the mode expansion as

$$\phi(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=0}^{\infty} \phi_n(x) e^{iny/R} \tag{10.13}$$

This results to cylindrical condition $\phi(x, y + 2\pi R) = \phi(x, y)$. The massless free field action in M^5 will reduces into the form

$$\hat{S}[\phi] = \frac{1}{2}\int d^4x \int dy \partial_A \phi(x,y)\partial^A \phi(x,y) \tag{10.14}$$

$$= \frac{1}{2}\int d^4x \int dy \frac{1}{2\pi R}\sum_{n,m}\Big\{\partial_\mu \phi_n(x) e^{iny/R}\partial^\mu \phi_m(x) e^{imy/R} - \left(\frac{in}{R}\right)\phi_n(x) e^{iny/R}\left(\frac{im}{R}\right)\phi_m(x) e^{imy/R}\Big\} \tag{10.15}$$

$$= \sum_{n,m}\int d^4x \underbrace{\left(\frac{1}{2\pi R}\int dy e^{i(n+m)y/R}\right)}_{=\delta_{n,-m}} \times \left[\partial_\mu \phi_n(x)\partial^\mu \phi_m(x) + \frac{nm}{R^2}\phi_n(x)\phi_m(x)\right]$$

$$= \int d^4x \left[\partial_\mu \phi_0(x)\partial^\mu \phi_0(x) + \sum_{n}^{\infty}\left(\partial_\mu \phi_n(x)\partial^\mu \phi_{-n}(x) - \frac{n^2}{R}\phi_n(x)\phi_{-n}(x)\right)\right] \tag{10.16}$$

On M^4 we have massless scalar field for the mode $\phi_0(x)$ and the *KK-tower* of massive scalar field with mass $m_n = \frac{n}{R}$. These mass tower becomes huge masses when $R \to 0\infty$ and become small when $R \to 0$

10.2.2 Compactification of vector field

For a vector field $A_A(x, y)$ on M^5 it compactification into $M^4 \times S^1$ is done as

$$A_M(x,y) = \sum_{n=0}^{\infty} A_{n,M}(x) e^{iny/R} \tag{10.17}$$

And

$$\hat{S}[A] = \int d^4x \int dy \left[-\frac{1}{4}F_{MN}F^{MN}\right] \tag{10.18}$$

$$= \int d^4x \int dy \left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu A_4 - \partial_4 A_\mu\right)\left(\partial^\mu A^4 - \partial^4 A^\mu\right)\right] \tag{10.19}$$

$$= \int d^4x \sum_{n=0}^{\infty}\left[\frac{1}{4}F_{n,\mu\nu}F^{\mu\nu}_{-n} + \frac{1}{2}\left(\partial_\mu A_{-n,4} + \frac{in}{R}\partial_4 A_{-n,\mu}\right) \times \left(\partial^\mu A^4_n - \frac{in}{R}\partial^4 A^\mu_n\right)\right] \tag{10.20}$$

Apply with the following gauge transformations

$$A_n^\mu(x) \to A_n^\mu(x) - \frac{i}{n/R} A_n^4, \ A_n^4 \to 0, \text{ for } n \neq 0 \tag{10.21}$$

Then we have

$$\begin{aligned} S[A] &= \int d^4x \left[-\frac{1}{4} F_{0,\mu\nu} F_0^{\mu\nu} + \frac{1}{2} \partial_\mu A_0^4 \partial^\mu A_0^4 \right] \\ &+ \int d^4x \sum_{n\geq 1}^{\infty} \left[-\frac{1}{4} F_{-n,\mu\nu} F_n^{\mu\nu} + \frac{1}{2} \frac{n^2}{R} A_{-n,\mu} A_n^\mu \right] \end{aligned} \tag{10.22}$$

We also get a *KK tower* of massive vector fields of the modes $n \geq 1$ and massless vector field and massless scalar field A_0^4 for the mode $n = 0$.

The matter-coupled gauge field can be constructed with the covariant derivative defined on M^5 in the form

$$D_M = \partial_M + ig_5 A_M, \ g_5 - \text{gauge coupling in d} = 5 \tag{10.23}$$

$$\mapsto D_\mu = \partial_\mu + ig_5 A_\mu \tag{10.24}$$

$$\partial_\mu + ig_5 \frac{1}{\sqrt{2\pi R}} A_{0,\mu} + ... \tag{10.25}$$

This shows that

$$g_4 = \frac{g_5}{\sqrt{2\pi R}} \tag{10.26}$$

where g_4 grows up (strong coupling) as $R \to 0$ and dies out (weak coupling) as $R \to \infty$.

10.3 Extradimensions

There are models of extra dimension beyond $4 + 1$. which seem to have applications in gravity and particle physics.

10.3.1 Warped (small) extra dimension

Randall–Sundrum model

10.3.2 Large extra dimension

ADD model, by Nima Arkani-Hamed, Savas Dimopoulos, and Gia Dvali

10.3.3 Universal extra dimension

It is just RS + ADD model