11 Conformal Symmetry and Conformal Field

11.1 Conformal group and algebra

A conformal transformation of space-time coordinate on $\mathcal{M}^4$ is defined as

$$g_{\mu\nu} \xrightarrow{CF} g'_{\mu\nu} = \frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\nu}{\partial x_\beta} g_{\alpha\beta} \equiv \Omega(x) g_{\mu\nu} \tag{11.1}$$

Where $\Omega(x)$ is conformal factor and it is determined from infinitesimal transformation as

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{11.2}$$

$$\mapsto \frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu_\alpha + \partial_\alpha \epsilon^\mu(x) \tag{11.3}$$

$$\begin{aligned} \mapsto \frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\nu}{\partial x_\beta} g_{\alpha\beta} &= g_{\alpha\beta}(\delta^\alpha_\mu + \partial^\alpha \epsilon_\mu)(\delta^\beta_\nu + \partial^\beta \epsilon_\nu) \\ &= g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \end{aligned} \tag{11.4}$$

Define

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = K g_{\mu\nu} \tag{11.5}$$

$$\mapsto g^{\mu\nu}\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = K g^{\mu\nu} g_{\mu\nu} \to 2(\partial \cdot \epsilon) = dK \tag{11.6}$$

$$\mapsto \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu} \tag{11.7}$$

$$\mapsto (d-1)\partial_\mu \partial^\mu (\partial \cdot \epsilon) = 0 \tag{11.8}$$

with $d = 4 = dim(\mathcal{M}^4)$. These become constrain conditions of the conformal transformation (15.2).

The generic form of $\epsilon^\mu(x)$ is

$$\epsilon^\mu(x) = a^\mu + b^{\mu\nu} x_\nu + c^{\mu\nu\sigma} x_\nu x_\sigma \tag{11.9}$$

The first term is known as *translation*, the second term is known as *dilation and rotation*, while the third term is known as *special conformal transformation.*

11.1.1 Translation

When $\epsilon^\mu = a^\mu$ is a constant translation, we observe nothing from (15.7) that $0 = 0$. But we already know that the generator of translation is the momentum operator

$$P_\mu = i\partial_\mu \tag{11.10}$$

11.1.2 Dilation and rotation

When $\epsilon^\mu = b^{\mu\nu}x_\nu$, we can observe from (15.7) that

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\rho{}_\rho g_{\mu\nu} \tag{11.11}$$

$$\xrightarrow{decompose} b_{\mu\nu} = \alpha g_{\mu\nu} + \tilde{b}_{[\mu\nu]} \tag{11.12}$$

where α is *dilation factor* and $\tilde{b}_{[\mu\nu]}$ is the anti-symmetric tensor for rotations. Note that dilation is generated by *Dilation operator* D, while rotation is generated by *angular momentum tensor* $M_{\mu\nu}$.

11.1.3 Special conformal transformation

When $\epsilon^\mu = c^{\mu\nu\rho}x_\nu x_\rho$, we can observe from (15.7) as

$$c^{\mu\nu\rho} = g^{\mu\rho}c^{\nu\sigma}{}_\sigma + g^{\mu\nu}c^{\rho\sigma}{}_\sigma - g^{\nu\rho}c^{\mu\sigma}{}_\sigma \tag{11.13}$$

11.2 Conformal symmetry of fields

11.3 Conformal field theory