8 Gravitational Field

8.1 General theory of relativity

8.1.1 Equivalence principle

According to the fact that

"*all observer cannot observe gravity from any inertial frame*"

This made Einstein to though about equivalence between non-inertial (accelerating) frame on flat space to the inertial frame in curved space Since

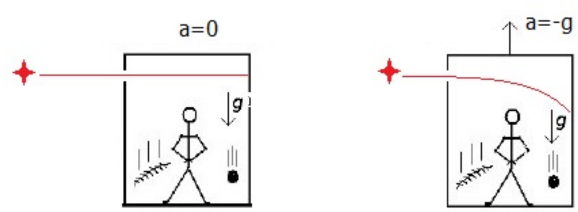

Figure 8.1: Einstein's equivalence principle.

light do not feel gravity and it travels along geometry of spacetime. We can observe light path in equivalence accelerating frame, without gravity, that the spacetime is curved. This principle makes us to study gravity from geometry of spacetime.

8.2 Geometry of curved spacetime

Let M be any 4-dimensional curved spacetime manifold equipped with metric tensor $g_{\mu\nu}(x)$ at any point P, with coordinate patch x^{μ} on tangent manifold TM_P.

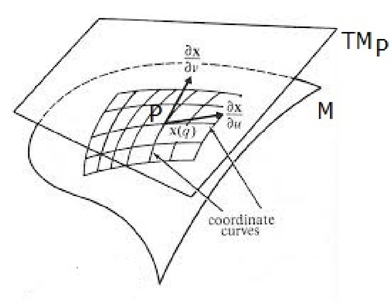

Figure 8.2: Coordinate patch of M on TM_P.

8.2.1 Geodesic equation and the connection

The geometry of M is determined from *geodesic path* trace on M from a to b, with path length

$$\tau = \int_a^b ds = \int_a^b \sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu} \equiv \int_a^b \sqrt{g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau \tag{8.1}$$

After we have parametrized the geodesic path with its path length τ. The shortest path can be determined from calculus of variation. Let

$$L = L(x,x') = \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}},\ x'^\mu = \frac{dx^\mu}{d\tau} \tag{8.2}$$

$$\begin{aligned} \delta\tau = 0 &= \int_a^b \left(\frac{\partial L}{dx^\mu}\delta x^\mu + \frac{\partial L}{\partial x'^\mu}\delta x'^\mu\right) d\tau \\ &= \int_a^b d\tau \left(\frac{\partial L}{dx^\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial x'^\mu}\right)\delta x^\mu d\tau + \int_a^b d\left(\frac{\partial L}{\partial x'^\mu}\delta x^\mu\right) \end{aligned} \tag{8.3}$$

The last term is assumed to be zero at boundary, thus we get the *Euler-Lagrange equation* of geodesic path in the form

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial x'^\mu} = 0 \tag{8.4}$$

$$\text{Let } L = \sqrt{F} \neq F(\tau) \mapsto \frac{\partial F}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial F}{\partial x'^\mu} = 0 \tag{8.5}$$

With $F(x,x') = g_{\mu\nu}(x)x'^\mu x'^\nu$, let us determine

$$\frac{\partial F}{\partial x^\rho} = g_{\mu\nu,\rho}x'^\mu x'^\nu,\ g_{\mu\nu,\rho} = \frac{dg_{\mu\nu}(x)}{dx^\rho} \tag{8.6}$$

$$\frac{\partial F}{\partial x'^\rho} = g_{\rho\nu}x'^\nu + g_{\mu\rho}x'^\mu \tag{8.7}$$

$$\mapsto \frac{d}{d\tau}\frac{\partial F}{\partial x'^\rho} = g_{\rho\nu,\mu}x'^\mu x'^\nu + g_{\mu\rho,\nu}x'^\mu x'^\nu + 2g_{\rho\mu}x''^\mu \tag{8.8}$$

From Euler-Lagrange equation, we have

$$2g_{\rho\mu}x''^\mu + \left(g_{\rho\nu,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho}\right)x'^\mu x'^\nu = 0 \tag{8.9}$$

$$\mapsto x''^\sigma + \frac{1}{2}g^{\sigma\rho}\left(g_{\rho\nu,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho}\right)x'^\mu x'^\nu = 0 \tag{8.10}$$

We can define the *Christoffel symbols* of the connection as

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(g_{\rho\nu,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho}\right) \tag{8.11}$$

$$(8.10) \mapsto x''^\sigma + \Gamma^\sigma_{\mu\nu}x'^\mu x'^\nu = 0 \tag{8.12}$$

It is called *geodesic equation* of the geodesic path on M.

8.2.2 Parallel transport

The*covariant derivative* of any vector V^β defined on M is defined the form

$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\gamma\alpha} V^\gamma \tag{8.13}$$
$$\text{and } \nabla_\alpha V_\beta = \partial_\alpha V_\beta - \Gamma^\gamma_{\beta\alpha} V_\gamma \tag{8.14}$$

It measures the change of V^β with it is traced along any geodesic path on M.

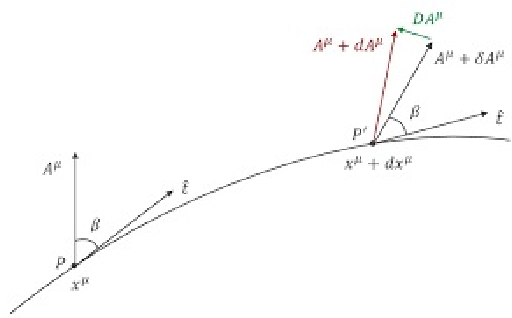

Figure 8.3: Change of any vector when parallel transported along geodesic path on M.

8.2.3 Riemann curvature tensor

The curvature of M can be measured by doing a parallel transport of any vector along a closed geodesic path An Riemann curvature tensor is defined in the form

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta \tag{8.15}$$

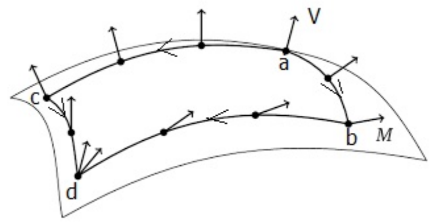

Figure 8.4: Parallel transport of a vector along a closed geodesic path on M.

Let us determine

$$[\nabla_\mu, \nabla_\nu]V^\alpha = \nabla_\mu \nabla_\nu V^\alpha - \nabla_\nu \nabla_\mu V^\alpha \tag{8.16}$$

$$\nabla_\mu \nabla_\nu V^\alpha = \nabla_\mu \left(\partial_\nu V^\alpha + \Gamma^\alpha_{\gamma\nu} V^\gamma\right) \tag{8.17}$$

$$= \partial_\mu \partial_\nu V^\alpha + \Gamma^\alpha_{\gamma\nu,\mu} V^\gamma + \Gamma^\alpha_{\gamma\nu} \partial_\mu V^\gamma + \Gamma^\alpha_{\gamma\mu} \partial_\nu V^\gamma + \Gamma^\alpha_{\delta\mu} \Gamma^\delta_{\gamma\nu} V^\gamma \tag{8.18}$$

$$\nabla_\nu \nabla_\mu V^\alpha = \partial_\nu \partial_\mu V^\alpha + \Gamma^\alpha_{\gamma\mu,\nu} V^\gamma + \Gamma^\alpha_{\gamma\mu} \partial_\nu V^\gamma + \Gamma^\alpha_{\gamma\nu} \partial_\mu V^\gamma + \Gamma^\alpha_{\delta\nu} \Gamma^\delta_{\gamma\mu} V^\gamma \tag{8.19}$$

$$(8.16) \mapsto [\nabla_\mu, \nabla_\nu]V^\alpha = \left(\Gamma^\alpha_{\gamma\nu,\mu} - \Gamma^\alpha_{\gamma\mu,\nu} + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\gamma\nu} - \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\gamma\mu}\right) V^\gamma \tag{8.20}$$

Then we have an expression of Riemann curvature tensor in the form

$$R^\alpha{}_{\gamma\mu\nu} = \Gamma^\alpha_{\gamma\nu,\mu} - \Gamma^\alpha_{\gamma\mu,\nu} + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\gamma\nu} - \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\gamma\mu} \tag{8.21}$$

Basic properties of Riemann curvature tensor are

- $R_{\alpha\gamma\mu\nu} = R_{\mu\nu\alpha\gamma}$
- $R_{\alpha\gamma\mu\nu} = -R_{\gamma\alpha\mu\nu} = -R_{\alpha\gamma\nu\mu}$
- $R_{\alpha\gamma\mu\nu} + R_{\alpha\mu\nu\gamma} + R_{\alpha\nu\gamma\mu} = 0$
- $\nabla_\beta R_{\alpha\gamma\mu\nu} + \nabla_\alpha R_{\gamma\beta\mu\nu} + \nabla_\gamma R_{\beta\alpha\mu\nu} = 0$ (Bianchi's identity)

8.3 Einstein equation

The Ricci tensor is defined from the contraction as

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} \tag{8.22}$$

And Ricci scalar is defined by another contraction

$$R = g^{\mu\nu} R_{\mu\nu} \tag{8.23}$$

Einstein tensor is then defined from Ricci tensor and Ricci scalar in the form

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{8.24}$$

And Einstein wrote his equation of gravity, according to equivalence principle, in the form

$$E_{\mu\nu} = \kappa T_{\mu\nu} \tag{8.25}$$

where κ is Einstein's proportional constant and $T_{\mu\nu}$ is *stress-energy tensor* of matter field. When contraction both side with metric tensor, we will have

$$g^{\mu\nu} E_{\mu\nu} = R - \frac{1}{2} \delta^{\mu}_{\mu} R = -R = \kappa T \tag{8.26}$$

$$(8.24) \mapsto R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{8.27}$$

In free space, we will have Einstein's equation in the form

$$R_{\mu\nu} = 0 \tag{8.28}$$

8.3.1 Newtonian limit

From geodesic equation (8.12), in low energy limit only the temporal component of velocity $x'^{\mu} \sim (u^0, 0, 0, 0)$ is dominate, then we will have

$$x''^{\mu} + \Gamma^{\mu}_{00} u^0 u^0 = 0 \text{ and } \Gamma^{\mu}_{00} \sim -\frac{1}{2} g^{\mu\nu} \partial_{\nu} g_{00} \tag{8.29}$$

Let us assume weak gravity as $g^{\mu\nu} \sim \eta_{\mu\nu} - h_{\mu\nu}$, so that

$$\Gamma^{\mu}_{00} \sim \frac{1}{2} g^{\mu\nu} \partial_{\nu} h_{00} \mapsto x''^{\mu} \sim \frac{1}{2} g^{\mu\nu} \partial_{\nu} h_{00} = \frac{1}{2} \eta^{ij} \partial_j h_{00} = -\partial^i h_{00} \tag{8.30}$$

Note that

$$T_{00} = \rho = T \xrightarrow{(8.27)} R_{00} = \frac{1}{2} \kappa \rho \tag{8.31}$$

$$\text{Since } R_{00} \sim \partial_i \Gamma^{i}_{00} = -\frac{1}{2} \partial_i \partial^i h_{00} = \nabla^2 \Phi, \ \Phi = \frac{1}{2} h_{00} \tag{8.32}$$

$$\mapsto \nabla^2 \Phi = \frac{1}{2} \kappa \rho \equiv 4\pi G \rho \tag{8.33}$$

where G is Newton's gravitational constant. Then we have

$$\kappa = 8\pi G \to \frac{8\pi}{c^4} G \tag{8.34}$$

8.4 Einstein-Hilbert action

Einstein's equation(8.27), with matter coupled gravity, can be derived from least action principle. We start from Einstein-Hilbert action in the form

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa}R + \mathcal{L}_{matter}\right) \equiv S_{EH} + S_M \tag{8.35}$$

Note that $\sqrt{-g}d^4$ is *diffeomorphism invariant* integral measure on M, i.e.,

$$d^4x' = \left|\frac{dx'}{dx}\right| d^4x, \text{ and } g_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'_\nu}g_{\alpha\beta}(x)$$

$$\mapsto g' = \left|\frac{dx}{dx'}\right|\left|\frac{dx}{dx'}\right| g, \text{and } \sqrt{-g'} = \left|\frac{dx}{dx'}\right|\sqrt{-g}$$

$$\mapsto \sqrt{-g'}d^4x' = \sqrt{-g}d^4x$$

Apply with least action principle

$$\Delta S = \delta S_{EH} + \delta S_M = 0 \tag{8.36}$$

where

$$\delta S_{EH} = \frac{1}{2\kappa}\int d^4x\left(\delta(\sqrt{-g})R + \sqrt{-g}\delta R\right) \tag{8.37}$$

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}}\frac{dg}{dg_{\alpha\beta}}\delta g_{\alpha\beta} \tag{8.38}$$

Since

$$g^{\alpha\beta} = (g_{\alpha\beta})^{-1} = \frac{1}{g}(G^{\alpha\beta})^T = \frac{1}{g}G^{\beta\alpha},\ G^{\alpha\beta} = \text{co} - \text{factor of } g_{\alpha\beta}$$

$$\mapsto g = g_{\alpha\beta}G^{\beta\alpha}$$

Then we have from above

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}G^{\beta\alpha}\delta g_{\alpha\beta} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta} \tag{8.39}$$

$$R = g^{\alpha\beta}R_{\alpha\beta} \mapsto \delta R = R_{\alpha\beta}\delta g^{\alpha\beta} + g^{\mu\nu}\delta R_{\mu\nu} \tag{8.40}$$

From (8.21)

$$\delta R^\alpha{}_{\gamma\mu\nu} = \delta\Gamma^\alpha_{\gamma\nu,\mu} - \delta\Gamma^\alpha_{\gamma\mu,\nu} + \delta\Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\gamma\nu} + \Gamma^\alpha_{\delta\mu}\delta\Gamma^\delta_{\gamma\nu} - \delta\Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\gamma\mu} - \Gamma^\alpha_{\delta\nu}\delta\Gamma^\delta_{\gamma\mu}$$

$$\equiv \nabla_\mu \Gamma^\alpha_{\gamma\nu} - \nabla_\nu \Gamma^\alpha_{\gamma\mu}$$
$$\mapsto \delta R^\alpha{}_{\mu\alpha\nu} = \nabla_\alpha \Gamma^\alpha_{\mu\nu} - \nabla_\alpha \Gamma^\alpha_{\nu\mu} = 0$$

Then we have from above

$$\delta R = R_{\alpha\beta}\delta g^{\alpha\beta} \tag{8.41}$$
$$\mapsto \delta S_{EH} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)\delta g^{\alpha\beta} \tag{8.42}$$
$$\delta S_M = \int d^4x\delta\left(\sqrt{-g}\mathcal{L}_{matter}\right) = -\frac{1}{2}\int d^4\sqrt{-g}T_{\alpha\beta}\delta g^{\alpha\beta} \tag{8.43}$$
$$\text{where } T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\partial\sqrt{-g}\mathcal{L}_{matter}}{\partial g^{\alpha\beta}} \tag{8.44}$$

From (8.36), we can derive the Einstein's equation in the form

$$\delta S = 0 = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R - \kappa T_{\alpha\beta}\right)\delta g^{\alpha\beta} \tag{8.45}$$
$$\mapsto R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa T_{\alpha\beta} \tag{8.46}$$

8.5 Schwarzschild static solution

Trial static solution of Einstein equation in free isotropic space is proposed by Schwarzschild in the form

$$ds^2 = U(r)(dt)^2 - V(r)(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2 \tag{8.47}$$

This corresponds to the metric tensor

$$\mapsto g_{00} = U(r), g_{11} = -V(r), g_{22} = -r^2,\ g_{33} = -r^2\sin^2\theta \tag{8.48}$$
$$g^{00} = \frac{1}{U(r)},\ g^{11} = -\frac{1}{V(r)},\ g^{22} = -\frac{1}{r^2},\ g^{33} = -\frac{1}{r^2\sin^2\theta} \tag{8.49}$$

Use these information to calculate the connection and then Riemann curvature tensor, Ricci tensor and Ricci scalar. After insertion into Eintein equation (8.28), we can solve form $U(r)$ and $V(r)$, and we can find that

$$U(r) = 1 - \frac{C}{r},\ V(r) = \frac{1}{1-\frac{C}{r}} \tag{8.50}$$

From dimensional analysis, we can assign the value of constant of integration $C = \frac{2GM}{r}$, in natural unit. From (8.47) we will have

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2GM}{r}} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{8.51}$$

This solution shows us two singular points in radial direction, at $r = 0$ and at $r = r_0 = 2GM$. The geometry changes from regular spacetime at $r > r_0$ to non-regular spacetime at $0 < r < r_0$. The non-regular spacetime is said to be inside the *balck hole*, where $r = r_0$ is its *event horizon*. It is called *Schwarzschild radius*. So that any physical object cannot pass through the event horizon, event the light, and vice versa.

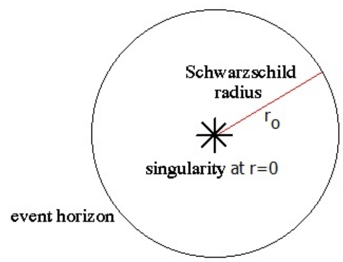

Figure 8.5: Schwarzschild black hole.

8.6 Linearized gravity

In the weak gravitational field we can assume the *linearized gravity* as

$$g_{\mu\nu}(x) \simeq \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{8.52}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric tensor. With this assumption we can calculate the connection, up to the first order of $h_{\mu\nu}(x)$, in the form

$$\Gamma^{\alpha}_{\mu\nu} \simeq \frac{1}{2}\eta^{\alpha\beta}\left(h_{\beta\nu,\mu} + h_{\beta\mu,\nu} - h_{\mu\nu,\beta}\right) \tag{8.53}$$

And the Riemann curvature tensor is calculated up to the first order in $h_{\mu\nu}(x)$ in the form

$$\begin{aligned} R^{\alpha}{}_{\rho\mu\nu} \simeq \Gamma^{\alpha}_{\rho\nu,\mu} - \Gamma^{\alpha}_{\rho\mu,\nu} &= \frac{1}{2}\eta^{\alpha\beta}\left(h_{\beta\nu,\rho\mu} + h_{\beta\rho,\nu\mu} - h_{\rho\nu,\beta\mu}\right. \\ &\qquad \left. -h_{\beta\mu,\rho\nu} - h_{\beta\rho,\mu\nu} + h_{\rho\mu,\beta\nu}\right) \\ &= \frac{1}{2}\eta^{\alpha\beta}\left(h_{\beta\nu,\rho\mu} + h_{\rho\mu,\beta\nu} - h_{\rho\nu,\beta\mu} - h_{\beta\mu,\rho\nu}\right) \end{aligned} \tag{8.54}$$

$$\begin{aligned} \mapsto R_{\mu\nu} &\simeq \frac{1}{2}\eta^{\alpha\beta}\left(h_{\beta\nu,\mu\alpha} + h_{\mu\alpha,\beta\nu} - h_{\mu\nu,\beta\alpha} - h_{\beta\alpha,\mu\nu}\right) \\ &= \frac{1}{2}\left(\partial_\mu\partial_\alpha h^{\alpha}_{\nu} + \partial_\nu\partial_\alpha h^{\alpha}_{\mu} - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h\right) \end{aligned} \tag{8.55}$$

We can assign a condition

$$\partial_\alpha h^\alpha_\mu - \frac{1}{2}\partial_\mu h = 0 \tag{8.56}$$

It is called *harmonic gauge*, then we have from above (8.55) and Einstein equation in free space (8.28)

$$\partial^2 h_{\mu\nu}(x) = 0 \tag{8.57}$$

8.6.1 Gravitational wave solution

A trial plane wave solution of (8.57) is

$$h_{\mu\nu}(x) \sim \epsilon_{\mu\nu}(k,\lambda)a(k,\lambda)e^{-ik\cdot x} \mapsto -k^2\epsilon_{\mu\nu}(k\lambda)a(k,\lambda) = 0 \tag{8.58}$$

$$k^2 = \omega^2 - |\vec{k}|^2 = 0 \mapsto \omega^2 - \omega_k^2 = 0, \omega_k = |\vec{k}| \tag{8.59}$$

Its general solution is

$$h_{\mu\nu}(x) = \int \frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\sum_\lambda \epsilon_{\mu\nu}(k,\lambda)\left(a(k,\lambda)e^{-ik\cdot x} + c.c.\right) \times(2\pi)\delta(\omega^2-\omega_k^2)\theta(\omega) \tag{8.60}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_\lambda \epsilon_{\mu\nu}(k,\lambda)\left(a(k,\lambda)e^{-ik\cdot x} + c.c.\right)_{\omega=\omega_k} \tag{8.61}$$

Since gravitational wave travels at light speed. And according to the harmonic gauge condition the polarization tensor must be symmetric, transversal and traceless. Let $k^\mu = (\omega_k, 0, 0, k)$, then we should have it in the form

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_{11} & \epsilon_{12} & 0 \\ 0 & \epsilon_{12} & -\epsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8.62}$$