0 Introduction

0.1 Natural unit

We will work in unit in which light speed and Plank constant are measured to be $c = 1 = \hbar$. In this unit we will observe basic physical quantities in dimension of mass

$$[mass] = [energy] = [momentum] = [length]^{-1} = [time]^{-1}$$

0.2 Minkowski space

The formulation will be done on four-dimensional Minkowski space $\mathcal{M}^4$, with coordinates

$$x \in \mathcal{M}^4 \mapsto x = x^\mu e_\mu,\ x = x_\mu e^\mu,\ e^\mu e_\nu = \delta^\mu \nu$$

$$g_{\mu\nu} = e_\mu e_\nu,\ g^{\mu\nu} = (g_{\mu\nu})^{-1} \mapsto g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu \nu, g_{\mu\nu} := diag.(+,-,-,-)$$

$$\mapsto x^\mu = (x^0, x^1, x^2, x^3), x_\mu = (x_0, x_1, x_2, x_3) = g_{\mu\nu} x^\nu = (x^0, -x^1, -x^2, -x^3)$$

$$x^2 = x^\mu x_\nu = g_{\mu\nu} x^\mu x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

$$\partial_\mu = (\partial_0, \nabla), \partial^\mu = (\partial_0, -\nabla) \mapsto \partial^2 = \partial_\mu \partial^\mu = \partial_0^2 - \nabla^2$$

0.3 Special relativity

Special relativity (SR) can be determined though Lorentz transformation (LT) of 4-position $x^\mu = (t, \vec{x})$, i.e. $x^\mu \in \mathcal{M}^4$, which is written in the form

$$x^\mu \xrightarrow{LT} x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

For example the case of Lorentz boost in 01-plane, we have

$$\Lambda^\mu{}_\nu \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{matrix} t' = \gamma(t - \beta x), \\ x' = \gamma(x - \beta t), \\ y' = y, z' = z \end{matrix}$$

An infinitesimal LT is written in the form

$$\Lambda^\mu{}_\nu \simeq \delta^\mu_\nu + \omega^\mu{}_\nu,\ \text{where } \omega^{\mu\nu} = -\omega^{\nu\mu}$$

0.4 Lorentz group, algebra and representation

From SR we know that spacetime distance squared is Lorentz scalar

$$x'^2 = t'^2 - \vec{x}' \cdot \vec{x}' = x^2 = t^2 - \vec{x} \cdot \vec{x} \equiv \tau^2 \text{ when } \vec{x} = 0$$

where τ is a proper time. From the viewpoint of LT we observe that

$$x'^2 = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma = g_{\rho\sigma} x^\rho x^\sigma = x^2$$

$$\mapsto g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma} \to (\det\Lambda)^2 = 1$$
$$1 = (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2 \mapsto (\Lambda^0{}_0)^2 \geq 1$$
$$\text{Or } \Lambda^0{}_0 \geq= +1, \text{ and } \Lambda^0{}_0 \leq -1$$

The LT Λ with $\det\Lambda = +1$ and $\Lambda^0{}_0 \geq= +1$ form to be a continuous transformation known as *proper othochronous Lorentz transformation* or SO(3,1). When LT is a symmetry transformation SO(3,1) will be a symmetry group. The generator of this group is determined from its infinitesimal transformation.

Let $f(x)$ be Lorentz scalar function, i.e. invariant in form under LT, such that

$$x \xrightarrow{LT} x', f(x) \xrightarrow{LT} f'(x') = f(x) \mapsto f'(x) = f(\Lambda^{-1}x)$$
$$x'^{\mu} \simeq x^{\mu} - \omega^{\mu\nu}x_{\nu} \mapsto f'(x) = f(x - \omega x) = f(x) - \omega^{\mu\nu}x_{\nu}\partial_{\mu}f(x) + ...$$
$$f'(x) = f(x) - \frac{1}{2}\omega^{\mu\nu}(x_{\nu}\partial_{\mu} - x_{\mu}\partial_{\nu})f(x) + ...$$
$$= \left(1 - \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} + ...\right)f(x) = e^{-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}f(x)$$
$$\text{when } M_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$$

This sows that $M_{\mu\nu}$ is a generator of LT, it satisfies the algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = ig_{\mu\rho}M_{\nu\sigma} + ig_{\nu\sigma}M_{\mu\rho} - ig_{\mu\sigma}M_{\nu\rho} - ig_{\nu\rho}M_{\mu\sigma}$$

Since LT is rotations on spatial plane and spacetime plane (boost), so that we can define

$$M_{0i} = K_i,\ M_{ij} = \frac{1}{2}\epsilon_{ijk}J_k$$
$$\mapsto [K_i, K_j] = i\epsilon_{ijk}J_k,\ [K + i, J_j] = -i\epsilon_{ijk}K_k,\ [J_i, J_j] = i\epsilon_{ijk}J_k$$

We observe two coupled rotations, in order to decoupling them we can do linear combination

$$J_i^{\pm} = \frac{1}{2}(J_i \pm iK_i) \mapsto [J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk}J^{\pm}k,\ [J_i^{\pm}, J_j^{\mp}] = 0$$

So that two Casimir operators of the Lorentz algebra are constructed from these two generalized angular momentum $J^{\pm}$

$$C_1 = (J^+)^2 = j_1(j_1 + 1),\ C_2 = (J^-)^2 = j_2(j_2 + 1)$$
$$j_1, j_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...$$

and the Lorentz representation is written in the form (j_1, j_2). The following is a list of some Lorentz representation:

(j_1, j_2)	Spin	Name	Symbol
$(0, 0)$	1	KG-scalar	ϕ
$(\frac{1}{2}, 0), (0, \frac{1}{2})$	$\frac{1}{2}$	L-Weyl spinor, R-Weyl spinor	$\psi_{\alpha}, \psi^{\dot{\alpha}}$
$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$	$\frac{1}{2}$	Dirac spinor	Ψ
$(\frac{1}{2}, \frac{1}{2})$	1, (0)	Maxwell vextor+gauge cond.	A^{μ}
$(\frac{1}{2}, \frac{1}{2} \otimes [(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$	$\frac{3}{2}, (\frac{1}{2})$	Rarita-Schwinger + cond.	Ψ^{μ}
$(1, 1)$	2	Einstein tensor + gauge cond.	$h_{\mu\nu}$