1 Classical Dynamics of Particle and Continuum

1.1 Dynamics of a point particle

Let $\{q\}$ be a set of degree of freedom of a particle, its dynamics is determined from the Lagrangian, using least action principle, as

$$L = L(q, \dot{(q)}) \mapsto S[q] = \int_a^b dtL$$

$$\delta S[q] = 0 = \int_a^b dt\delta L = \int_a^b dt \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right)$$

$$= \underbrace{\int_a^b dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q \right)}_{=0 \ on \ boundary} + \int_a^b dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right)\delta q$$

$$\mapsto \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \text{ Euler} - \text{Lagrange equation}$$

The conjugate momentum p is derived from the Lagrangian as

$$p = \frac{\partial L}{\partial \dot{q}}$$

The particle Hamiltonian is derived from Legendre transformation of the Lagrangian as

$$H(q,p) = p\dot{q} - L \mapsto L = p\dot{q} + H, \ S[q] = \int_a^b dt(p\dot{q} + H)$$

$$\delta S[q] = 0 = \int_a^b dt \left(p\delta\dot{q} + \dot{q}\delta p - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p \right)$$

$$= (p\delta q)_a^b + \int_a^b dt \left(-\dot{p}\delta q + \dot{q}\delta p - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p \right)$$

$$\mapsto \dot{q} - \frac{\partial H}{\partial p} = 0, \ \dot{p} + \frac{\partial H}{\partial q} = 0, \ \text{Hamilton}'\text{sequations}$$

Define Poisson's bracket

$$A = A(q,p), B = (B(q,p) \mapsto \left\{ \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} \right\}$$

We will observe that

$$\dot{q} = \{q, H\}, \ \dot{p} = \{p, H\}, \ \text{and } \{q,p\} = 1 \ \ \text{symplec structure of phase space}$$

and (q,p) is called *canonical coordinates* of phase space. Note that $H = H(q,p)$ will form to be closed curve on phase space for closed system (constant energy).

1.2 Dynamics of a continuum

Let us determine a linear spring chain of N-mass points m connected with spring k of length a. Its longitudinal dynamics is determined from the Lagrangian

$$L = \sum_{i=1}^{N} \frac{1}{2} m \dot{\eta}_i^2 - \frac{1}{2} k \sum_{i=1}^{N-1} (\eta_{i+1} - \eta_i)^2$$

EoM of the i^{th}-mass point, after using Euler-Lagrange equation is

$$m\ddot{\eta}_i - k[(\eta_{i+1} - \eta_i) - (\eta_i - \eta_{i-1})] = 0$$

Apply the continuity approximation

$$N \to \infty, a \to 0,\ Length = Na(fixed)$$

$$L = a \left(\sum_{i=1}^{N} \frac{1}{2} \frac{m}{a} \dot{\eta}_i^2 - \frac{1}{2} ka \sum_{i=1}^{N-1} \frac{1}{a^2} (\eta_{i+1} - \eta_i)^2 \right) = \frac{1}{2} \int_0^L dx \left(\rho \dot{\eta}^2 - \kappa \eta'^2 \right)$$

$$L = \int_0^L dx \mathcal{L},\ \mathcal{L} = \frac{1}{2} \rho \dot{\eta}^2 - \frac{1}{2} \kappa \eta'^2$$

After we have used the notations

$$\sum_{i=1}^{N} a = \int_0^L dx,\ \eta_i(t) \to \eta(t,x),\ \frac{m}{a} \rho,\ ka = \kappa,\ \frac{1}{a}(\eta_{i+1} - \eta_i) = \frac{d\eta}{dx} = \eta'$$

The EoM will appear in the form

$$\rho \ddot{\eta} - \kappa \eta'' = 0$$

which can derive from the Euler-Lagrange equation of the form

$$\frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\eta}} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \eta'} = 0$$

The linear spring chain becomes an elastic rod, with longitudinal vibration. For an anisotropic elastic body in three-dimension, with displacement vector $\vec{\eta} = (\eta_1, \eta_2, \eta_3)$, we will have

$$\mathcal{L} = \frac{1}{2} \rho \sum_i \dot{\eta}_i^2 - \frac{1}{2} \sum_{i,j} \kappa_{ij} \eta_i \eta_j$$

Its EoM is derived from Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \eta_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\eta}_i} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla \eta_i} = 0$$

The conjugate momentum and Hamiltonian is derived as

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\eta}} = \rho \dot{\eta} \mapsto \mathcal{H} = \pi \dot{\eta} - \mathcal{L} = \frac{1}{2\rho} \pi^2 + \frac{1}{2} \kappa \eta'^2$$

In case of Lorentz covariant field $F(x)$,i.e. Lorentz representative fields, we will have

$$\mathcal{L} = \mathcal{L}(F, \partial_\mu F),\ S[F] = \int d^4x\mathcal{L}$$

$$\delta S = 0 = \int d^4x \delta\mathcal{L} = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial F}\delta F + \frac{\partial \mathcal{L}}{\partial\partial_\mu F}\delta\partial_\mu F \right)$$

$$= \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\partial_\mu F}\delta F \right) + \int d^4x \left(\frac{\partial \mathcal{L}}{\partial F} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu F} \right)\delta F$$

$$\mapsto \frac{\partial \mathcal{L}}{\partial F} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu F} = 0$$

$$\pi = \frac{\partial \mathcal{L}}{\partial\partial_0 F} \mapsto \mathcal{H} = \pi\partial_0 F - \mathcal{L},\ H = \int d^3\mathcal{H}$$

1.3 Continuous symmetry and Noether's theorem

The theorem state that:

"*an invariant (of the action) under any continuous transformation will correspond with conserve quantity*"

1.3.1 Point particle symmetry

For generic Lagrangian $L = L(q, \dot{q})$, let $q \to q' = q + \delta q_\epsilon$, an invariant action reads

$$\delta_\epsilon S[q] = 0 = \int dt \delta_\epsilon L = \int dt \left(\frac{\partial L}{\partial q}\delta_\epsilon q + \frac{\partial L}{\partial \dot{q}}\delta_\epsilon q \right)$$

$$0 = \int dt \frac{d}{dt}\left(\frac{\partial L}{\partial q}\delta_\epsilon q \right) + \int dt \underbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right)}_{=0\ from\ EL\ eqn.}\delta_\epsilon q$$

$$J = \frac{\partial L}{\partial \dot{q}}\delta_\epsilon q \mapsto \frac{dJ}{dt} = 0$$

J is the conserve quantity and known in the name of *Noether's current*. For example of free particle with translation symmetry

$$L = \frac{1}{2}m\dot{q}^2,\ \delta_\epsilon q = a \mapsto \frac{J}{a} = m\dot{q}$$

1.3.2 Continuous system symmetry

For continuous system, let us determine Lorentz covariant field. In case of generic field $F(x)$ with Lagrangian density $\mathcal{L}(F, \partial_\mu F)$. Let

$$x \to x' = x + \delta_\epsilon x,\ \text{and}\ F \to F' = F + \delta_\epsilon F$$

$$\text{where}\ \delta_\epsilon F = \bar{\delta}_\epsilon F + \delta_\epsilon x^\mu \partial_\mu F$$

are symmetry transformations of the continuous system. Note that $\bar{\delta}_\epsilon F$ is the local change of field. Then we have

$$\delta_\epsilon S[F] = 0 = \delta_\epsilon \left(\int d^4x \mathcal{L} \right) = \int \delta_\epsilon (d^4x) \mathcal{L} + \int d^4x \delta_\epsilon \mathcal{L}$$

Let us determine $x \to x' = x + \delta_\epsilon x$, so that

$$d^4x' = \det \left| \frac{dx'}{dx} \right| d^4x = (1 + \partial_\mu \delta_\epsilon x^\mu + ...) d^4x \equiv d^2x + \delta_\epsilon d^4x$$

$$\mapsto \delta_\epsilon (d^4x) = (\partial_\mu \delta_\epsilon) x^\mu d^4x, \text{ and } \int \delta_\epsilon (d^4x) \mathcal{L} = \int d^4x (\partial_\mu \delta_\epsilon x^\mu) \mathcal{L}$$

For the field transformation $F \to F'' = F + \delta_\epsilon F$, we will have

$$\int d^4x \delta_\epsilon \mathcal{L} = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial F} \bar{\delta}_\epsilon F + \frac{\partial \mathcal{L}}{\partial \partial_\mu F} \bar{\delta}_\epsilon \partial_\mu F + \delta_\epsilon x^\mu \partial_\mu \mathcal{L} \right)$$

$$= \int d^4x \partial_\mu \left(\delta x^\mu \partial_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu F} \bar{\delta}_\epsilon F \right) + \int d^4x \left(\underbrace{\frac{\partial \mathcal{L}}{\partial F} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu F}}_{=0 \ by \ EoM} \right) \bar{\delta}_\epsilon F$$

$$\mapsto 0 = \int d^4x \left[\underbrace{\partial_\mu \delta^\mu_\nu \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L}}_{=\partial_\mu (\delta_\epsilon x^\mu \mathcal{L})} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial_\mu F} (\delta_\epsilon F - \delta_\epsilon x^\nu \partial_\nu F) \right) \right]$$

$$= \int d^4x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu F} \delta_\epsilon F - \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu F} \partial_\nu F - \delta^\mu_\nu \mathcal{L} \right) \delta_\epsilon x^\nu \right] \equiv - \int d^4x \partial_\mu J^\mu$$

$$J^\mu = \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu F} \partial_\nu F - \delta^\mu_\nu \mathcal{L} \right) \delta_\epsilon x^\nu - \frac{\partial \mathcal{L}}{\partial \partial_\mu F} \delta_\epsilon F$$

It is the *conserved Noether's current.*

In case of translation transformation of translation symmetric field, we have

$$\delta_\epsilon F = 0, \ \delta_\epsilon x^\mu = a^\mu \mapsto J^\mu = \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu F} \partial_\nu F - \delta^\mu_\nu \mathcal{L} \right) a^\nu = T^\mu{}_\nu a^\nu$$

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu F} \partial_\nu F - \delta^\mu_\nu \mathcal{L}$$

It is called *energy-momentum tensor.*

In case of Lorentz transformation of Lorentz covariant field, we have

$$x \to x' = \Lambda x \sim (1 + \omega) x = x + \omega x \mapsto \delta_\epsilon x^\mu = \omega^{\mu\nu} x_\nu$$

$$F(x) \to F'(x') = D(\Lambda) F(x) \sim D(1 + \omega) F(x) = I \cdot F(x) + \frac{1}{2} \omega \Sigma \cdot F(x)$$

$$\mapsto \delta_\epsilon F^a(x) = \frac{1}{2}\omega^{\mu\nu}\Sigma^{ab}_{\mu\nu}F^b$$

where $I \equiv I^{ab}$ is identity matrix. So that

$$\mapsto J^\mu = T^\mu{}_\nu \omega^{\nu\sigma}x_\sigma - \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\partial_\mu F^a}\omega^{\nu\sigma}\Sigma^{ab}_{\nu\sigma}F^b \equiv \frac{1}{2}\left(M^\mu{}_{\nu\sigma} + S^\mu{}_{\nu\sigma}\right)\omega^{\nu\sigma}$$

$$M^\mu{}_{\nu\sigma} = T^\mu{}_\nu x_\sigma - T^\mu{}_\sigma x_\nu \text{ (angular momentum tensor)}$$

$$S^\mu{}_{\nu\sigma} = -\frac{\partial\mathcal{L}}{\partial\partial_\mu F^a}\Sigma^{ab}_{\nu\sigma}F^b \text{ (spin} - \text{angular momentum tensor)}$$