2 Classical Dynamics of Scalar Fields

We will study classical dynamics of Klein-Gordon scalar field.

2.1 Klein-Gordon equation

Quantum dynamics can be determined from *canonical quantization* of classical system by promoting set of *canonical variables* (q, p), satisfy Poisson bracket $\{q, p\} = 1$, to be set of *canonical operator* $(\hat{q}, \hat{p})$ satisfy algebra (commutation relation) $[\hat{q}, \hat{p}] = i$. These operators have their own eigen-value equations

$$\hat{q}|q\rangle = q|q\rangle,\ \hat{p}|p\rangle = p|p\rangle$$

And their alternative applications are

$$\hat{q}|p\rangle = i\partial_p|p\rangle,\ \hat{p}|q\rangle = -i\partial_q|q\rangle \mapsto \langle q|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipq}$$

The system Hamiltonian $H = H(q, p)$ is also promoted to be an operator and has its own eigen-value equation

$$\hat{H}|\psi_E\rangle = E|\psi_E\rangle$$

which is known in the name of *Schrodinger's equation*. For non-relativistic equation we have

$$H = \frac{p^2}{2m} + V(q) \mapsto \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$$

$$\mapsto \left(\frac{\hat{p}^2}{2m} + V(\hat{q})\right)|\psi_E\rangle = E|\psi_E\rangle$$

$$\langle q|\left(\frac{\hat{p}^2}{2m} + V(\hat{q})\right)|\psi_E\rangle = \langle q|E|\psi_E\rangle \mapsto \left(-\frac{1}{2m}\partial_q^2 + V(q)\right)\psi_e(q) = E\psi_E(q)$$

when $\psi_E(q) = \langle q|\psi_E\rangle$. For relativistic particle, with relativistic energy-momentum relation

$$E^2 = p^2 + m^2 \mapsto E^2 - p^2 - m^2 = 0$$

Under canonical quantization, together with promoting an energy E to be an energy operator $\hat{E} = i\partial_t$ when viewd from time-dependent state $|\phi(t)\rangle$, we will have an operator equation

$$(\hat{E}^2 - \hat{p}^2 - m^2)|\phi(t)\rangle = 0 \mapsto -(\partial_t^2 - \partial_q^2 + m^2)\phi(t, q) = 0$$

when $\phi(t, q) = \langle q|\phi(t)\rangle$. In Lorentz covariant form $(t, q) = x^\mu$ and $\partial_t^2 - \partial_q^2 = \partial_\mu\partial^\mu = \partial^2$ is d'Alembertian which is Lorentz scalar. So that this equation can be Lorentz invariant when $\phi(x)$ is simply Lorentz scalar function. Then this equation become to know in the name of *Klein-Gordon equation*. To be quantum equation, the function $\phi(x)$ must be interpreted as probability density function, satisfy conserved probability condition. In case of complex scalar field, we have $\phi^* \neq \phi$ satisfy the same equation, and the probability current density is determined as

$$\phi^*(\partial^2 + m^2)\phi = 0 \xrightarrow[its\ conjugation]{subtraction\ with} \phi^*\partial^2\phi - \phi\partial^2\phi^* = 0$$

$$\mapsto \partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) \equiv \partial_\mu j^\mu = 0 \text{ continuity equation}$$

$$j^\mu\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^* = (\phi^*\partial_0\phi - \phi\partial_0\phi^*, -(\phi^*\nabla\phi - \phi\nabla\phi^*)) \equiv (j^0, \vec{j})$$

Note that $j^0 = \phi^*\partial_0\phi - \phi\partial_0\phi^*$ is not positive definite so that it cannot be interpreted as probability density. This is the fail point of Klein-Gordon equation for using as relativistic quantum equation. Another problem of Klein-Gordon equation is the negative energy part of quantum particle, since $E = \pm\sqrt{p^2 + m^2}$, in which Klein-Gordon cannot discuss this point in their time of writing their equation.

2.2 Classical aspects of scalar field

Reinterpretation of Klein-Gordon equation appear later as classical Lorentz scalar field equation, not a relativistic quantum equation any more. In case of real scalar field $\phi = \phi^*$, Klein-Gordon equation can be derived from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

From Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0 \mapsto \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi, \frac{\partial\mathcal{L}}{\partial_\mu\phi} = \partial^\mu\phi \mapsto -(\partial^2\phi + m^2\phi) = 0$$

Its Hamiltonian is

$$\pi = \frac{\partial\mathcal{L}}{\partial\partial_0\phi} = \partial_0\phi \mapsto \mathcal{H} = \phi\partial_0\phi - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{2}m^2\phi^2$$

Free field solution is determined from trial solution

$$\phi(x) \sim a(k)e^{-ik\cdot x} \mapsto (-k^2 + m^2)a(k) = 0$$

$$\mapsto -k^2 + m^2 = -\omega^2 + \vec{k}^2 + m^2 = 0 \mapsto \omega^2 = \omega_k^2,\ \omega_k = \sqrt{\vec{k}^2 + m^2}$$

after using notation of the 4-momentum $k^\mu = (\omega, \vec{k})$. The general solution is written in the form of Fourier transformation

$$\phi(x) = \int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\left(a(k)e^{-ikx} + a^*(k)e^{ikx}\right)(2\pi)\delta(\omega^2 - \omega_k^2)$$

Using identity of the delta function

$$\delta(f(x)) = \sum_i\frac{f(a_i)}{|f'(a_i)|},\ f(a_i) = 0 \text{ for all } i$$

$$\delta(\omega^2 - \omega_k^2) = \frac{1}{2\omega_k}[\delta(\omega - \omega_k) + \delta(\omega + \omega_k)]$$

The we have from above

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k)e^{-ikx} + a^*(k)e^{ikx}\right)_{\omega=\omega_k}$$

We also have

$$\pi = \partial_0 \phi = -\frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \left(a(k)e^{-ikx} - a^*(k)e^{ikx}\right)_{\omega=\omega_k}$$

$$\nabla\phi(x) = i \int \frac{d^3k}{(2\pi)^3 2\omega_k} \vec{k} \left(a(k)e^{-ikx} - a^*(k)e^{ikx}\right)_{\omega=\omega_k}$$

The expression of amplitude $a(k)$ can be derived from the inverse Fourier transformation as

$$a(k) = i \int d^3x e^{ikx}(\partial_0\phi(x) - i\omega_k\phi(x)) \equiv i \int d^3x e^{ikx} \overleftrightarrow{\partial}_0 \phi(x)$$

$$\mapsto a^*(k) = -i \int d^3x e^{-ikx} \overleftrightarrow{\partial}_0 \phi(x), \text{ when } \phi^* = \phi$$

after we have used the notation $f(x)\overleftrightarrow{\partial}_0 g(x) = f(x)\overrightarrow{\partial}_0 g(x) - f(x)\overleftarrow{\partial}_0 g(x)$.

From complex scalar field $\phi \neq \phi^*$, we have field Lagrangian in the form

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

The conjugate momentum field is

$$\pi = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} = \partial_0 \phi^* \mapsto \pi^* = \partial_0 \phi$$

and the field Hamiltonian is

$$\mathcal{H} = \pi \partial_0 \phi + \pi^* \partial_0 \phi^* - \mathcal{L} = \pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi$$

The general free field solution will appear in the form

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k)e^{-ikx} + b^*(k)e^{ikx}\right)_{\omega=\omega_k}$$

And the amplitudes $a(k)$ and $b(k)$ are derived as above. On the other hand the complex scalar field can be written in term of two real scalar fields as $\phi(x) = \phi_1(x) + i\phi_2(x) \mapsto \phi^*(x) = \phi_1(x) - i\phi_2(x) \neq \phi(x)$.

2.3 Conserved Noether's current

2.3.1 Energy-momentum tensor

Lorentz symmetry of the scalar field results to the conserved energy-momentum tensor and the angular momentum tensor of the form, in case of complex scalar field,

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*}\partial_\nu \phi^* - \delta^\mu \nu \mathcal{L} = \partial^\mu \phi^* \partial_\nu \phi - \delta^\mu \nu \mathcal{L}$$

$$Q = \int d^3x T^{00} = \int d^3x(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi) = \int d^3x \mathcal{H}$$

$$P^i = \int d^3x T^{0i} = \int d^3x(\partial^0 \phi^* \partial^i \phi + \partial^0 \phi \partial^i \phi^*)$$

These are the conserved Noether's charge Q (Hamiltonian H) and Noether's current J^i (momentum P^i).

2.3.2 Angular momentum tensor

The angular momentum tensor is

$$M^\mu{}_{\nu\sigma} = T^\mu{}_\nu x_\sigma - T^\mu{}_\sigma x_\nu \mapsto L_{ij} = M^0{}_{ij} = \int d^3x \left(T^0{}_i x_j - T^0{}_j x_i\right)$$

More analysis can be done on quantum level.

2.3.3 Spin angular momentum tensor

Since $\phi'(x') = D(\Lambda)\phi(x) = \phi(x) \mapsto D(\Lambda) = 1$ for scalar fields, so that

$$S^\mu{}_{\nu\sigma} = 0$$

This shows that the scalar fields have spin $s = 0$.

2.4 Self-interaction, ground state configuration and spontaneous symmetry breaking

2.4.1 Z_2-symmetry

In ϕ^4-self interaction model of the real scalar field, its Lagrangian appear in the form

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\mu^2 \phi^2 - \frac{1}{4} g \phi^4 \\ &\to \mathcal{V}(\phi) = \frac{1}{2}\mu^2 \phi^2 + \frac{1}{4} g \phi^4 \end{aligned}$$

This model has discrete Z_2-symmetry, i.e., $\phi \to -\phi$. The ground state of system is determined from the minimum of potential $\mathcal{V}(\phi = \phi_0)$. We observe that

$$\mathcal{V}'(\phi_0) = 0 = \mu^2 \phi_0 + g\phi_0^3 = \phi_0(\mu^2 + g\phi_0^2)$$

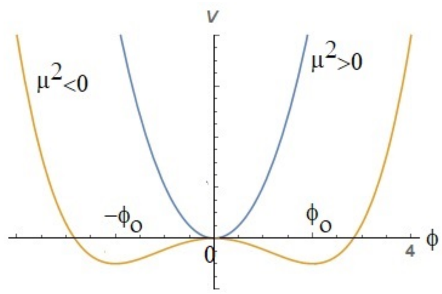

Figure 2.1: ϕ^4- potential.

$$\mu^2 > 0 \mapsto \phi_0 = 0, \text{ and } \mu^2 < 0 \mapsto \phi_0^{\pm} = \pm\sqrt{|\mu^2|/g}$$

In case of $\mu^2 > 0$, there is only one symmetric ground state configuration $\phi_0 = 0$, so there will be no symmetry breaking. But for $\mu^2 < 0$, there are Z_2-symmetric ground state configurations $\phi_0 = \pm\sqrt{|\mu|^2/g}$. The system must choose one of these result to the breaking of this symmetry, and this results to what is called *spontaneous symmetry breaking*. Let us choose one at $\phi_0 = +\sqrt{|\mu|^2/g}$, the field dynamics around this configuration can be determined from its fluctuation of the ground state as

$$\phi(x) \simeq \phi_0 + \eta(x)$$

Insertion into the field Lagrangian we will have

$$\partial_\mu \phi = \partial_\mu \eta, \;\; \phi^2 = (\phi_0 + \eta^2) = \phi_0^2 + 2\phi_0\eta + \eta^2 = \frac{|\mu|^2}{g} + 2\sqrt{\frac{|\mu|^2}{g}}\eta + \eta^2$$

$$\phi^4 = \phi_0^4 + 4\phi_0^3\eta + 6\phi_0^2\eta^2 + 4\phi_0\eta^3 + \eta^4$$
$$= \frac{|\mu|^4}{g^2} + 4\frac{|\mu|^3}{g\sqrt{g}}\eta + 6\frac{|\mu|^2}{g}\eta^2 + 4\sqrt{\frac{|\mu|^2}{g}}\eta^3 + \eta^4$$

Then the field Lagrangian after symmetry breaking will become

$$\mathcal{L} = -\frac{1}{4g}|\mu|^4 + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - |\mu|^2\eta^2 - \sqrt{g|\mu|^2}\eta^3 - \frac{1}{4}g\eta^4$$

The new system emerge with some ground state energy, differ in mass and extra cubic interaction.

2.4.2 SO(2)-symmetry

Let us determine complex scalar field $\phi = \phi_1 + i\phi_2$ with quartic interaction, its Lagrangian appear in the form

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - \mu^2\phi^*\phi - g(\phi^*\phi)^2$$

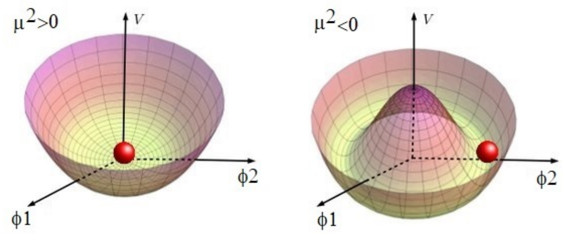

Figure 2.2: Ground state configuration of $(\phi^*\phi)^2$-interaction.

The ground state configuration appear as before but in two-dimensions.

In case of $\mu^2 > 0$, we see that

$$\phi_0^*\phi_0 = \phi_{01}^2 + \phi_{02}^2 = 0 \mapsto \phi_{01} = 0 = \phi_{02}$$

There are the symmetric ground state configurations. There will be no symmetry breaking. But in case of $\mu^2 < 0$, we see that

$$\phi_0^*\phi_0 = \phi_{01}^2 + \phi_{02}^2 = \frac{|\mu|^2}{2g}$$

We observe that ϕ_{01}, ϕ_{02} contribute to SO(2)-symmetric ground state configurations. There will be symmetry breaking. Let us break this symmetry by choosing the broken symmetry ground state configuration at

$$\phi_{01} = 0, \phi_{02} = a = +\sqrt{|\mu|^2/2g}, \mapsto \phi_1(x) \simeq \eta_1(x),\ \ \phi_2(x) \simeq a + \eta_2(x)$$

$$\partial_\mu\phi^*\partial^\mu\phi = \partial_\mu\eta_1\partial^\mu\eta_1 + \partial_\mu\eta_2\partial^\mu\eta_2$$

$$\phi^*\phi = \phi_1^2 + \phi_2^2 = \eta_1^2 + (a+\eta_2)^2 = \eta_1^2 + a^2 + 2a\eta_2 + \eta_2^2$$

$$(\phi^*\phi)^2 = \eta_1^4 + a^4 + 2a^2\eta_1^2 + 4a\eta_1^2\eta_2 + 2\eta_1^2\eta_2^2 + 6a^2\eta_2^2 + \eta_2^4 + 4a^3\eta_2 + 4a\eta_2^3$$

From the Lagrangian above we will have

$$\mathcal{L} = \partial_\mu\eta_1\partial^\mu\eta_1 + \partial_\mu\eta_2\partial^\mu\eta_2 + |\mu|^2\eta_1^2 + |\mu|^2\eta_2^2 + |\mu|^2a^2 + 2|\mu|^2a\eta_2$$

$$-g\eta_1^4 - ga^4 - 2ga^2\eta_1^2 - 4ga\eta_1^2\eta_2 - 2g\eta_1^2\eta_2^2 - 6ga^2\eta_2^2 - g\eta_2^4 - 4ga^3\eta_2 - 4a\eta_2^3$$

After insertion the value of s, we get important terms of the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \partial_\mu\eta_1\partial^\mu\eta_1 - g\eta_1^4 + \partial_\mu\eta_2\partial^\mu\eta_2 - 5|\mu|^2\eta_2^2 - g\eta_2^4 - \mathcal{L}_I(\eta_1, \eta_2)$$

We observe that η_1 becomes masssless scalar, where its mass is eaten by η_2 which get fat from mass $|\mu|^2 \to 5|\mu|^2$, and results with some constant energy $\mathcal{L}_0$ from the ground state configuration ϕ_{02} and coupling terms $\mathcal{L}_I$. The field η_1 is known in the name of *massless Goldstone's boson*, which is emerged from the process of *spontaneous symmetry breaking* of the SO(2) continuous symmetry of ground state configuration of the original model.

2.4.3 Goldstone's theorem

The theorem state that

"*breaking of any continuous symmetry of the degenerate ground state configuration spontaneously will result with massless boson*"

This is known as *Goldstone* or *Nambu-Goldstone boson*.