3 Classical Aspects of Spinor Fields

Spinor field is realized to be $(1/2, 0)$ and $(0, 1/2)$ representations of Lorentz group algebra. In this lecture we will determine their classical dynamics from Lagrangian and Hamiltonian descriptions.

3.1 Dirac spinor field

3.1.1 Dirac equation

Dirac spinor $Psi(x)$ was born from relativistic quantum equation wrote by Dirac. He start from linear relativistic energy-momentum relation

$$E = \vec{\alpha} \cdot \vec{p} + \beta m \tag{3.1}$$

which must be fulfill the quadratic relation

$$E^2 = (\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j + (\alpha_i \beta + \beta \alpha_i) p_i + \beta^2 m^2 \equiv p_i p_i + m^2 \tag{3.2}$$

$$\mapsto \alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}, \ \alpha_i \beta + \beta \alpha_i = 0, \ \beta^2 = 1 \tag{3.3}$$

Note that $\{\alpha_i\}$ and β cannot be numbers but square matrices. The smallest one are 2x2 matrices, i.e.,

$$\beta = \sigma^0, \ \alpha^i = \sigma^i, \ i = 1, 2, 3 \tag{3.4}$$

when

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that σ^0 is 2x2-identity, while $\{\sigma^i\}$ is a set of Pauli's matrices. These matrices can fulfill (3.3) except the second relation. Then Dirac move to 4x4 matrices, for convenient of their constructions, by defining

$$\beta = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \ \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \ i = 1, 2, 3 \tag{3.5}$$

These matrices zre completely fulfill (3.3). Then we get Dirac quantum equation, after doing the quantization by changing the some physical quantities to be quantum operators as

$$E \mapsto \hat{E} = i\partial_t, \ \vec{p} \mapsto -i\nabla$$

From (3.1), we will have

$$E - \vec{\alpha} \cdot \vec{p} - \beta m = 0 \mapsto (i\partial_t + i\vec{\alpha} \cdot \nabla - \beta m)\Psi(x) = 0 \tag{3.6}$$

Next let us define

$$\beta = \gamma^0, \ \beta\alpha^i = \gamma^i, \ i = 1, 2, 3 \mapsto \gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) \tag{3.7}$$

$$\mapsto \ \{\gamma^\mu, \gamma^\nu\} = 2g^{\alpha\beta}, \ \text{"Clifford algebra"} \tag{3.8}$$

With the fact that $\beta^2 = 1_{4x4}$, we have from (3.6)

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \mapsto (i\not{\partial} - m)\Psi(x) = 0 \tag{3.9}$$

where the existence of 1_{4x4} with m is understood, and $\not{\partial} = \gamma^\mu\partial_\mu$ is known as *Feynman slash* notation.

Let us define *Dirac conjugation* as

$$\bar{\Psi} = \Psi^\dagger\gamma^0 \mapsto -\Psi^\dagger(i(\gamma^\mu)^\dagger\overleftarrow{\partial}_\mu + m) = 0 \tag{3.10}$$

$$\Psi^\dagger\gamma^0(i\gamma^0(\gamma^\mu)^\dagger\gamma^0\overleftarrow{\partial}_\mu + m)\gamma^0 = 0,\ (\gamma^0)^2 = 1 \tag{3.11}$$

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \mapsto \bar{\Psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0 \tag{3.12}$$

We have derived the conjugation of Dirac equation. Multiply Dirac equation from the left with $\bar{\Psi}$ and multiply the conjugated equation from the right with Ψ, and do the summation we will get

$$\bar{\Psi}i\gamma^\mu\overleftarrow{\partial}_\mu\Psi + \bar{\Psi}\gamma^\mu\partial_\mu\Psi = \partial_\mu(i\bar{\Psi}\gamma^\mu\Psi) \equiv \partial_\mu j^\mu = 0 \tag{3.13}$$

$$j^\mu = i\bar{\Psi}\gamma^\mu\Psi \tag{3.14}$$

We have derived the conserved Dirac current density j^μ.

3.1.2 Lorentz invariant of Dirac equation

Let us determine Lorentz transformations of spacetime coordinate and Dirac spinor

$$x^\mu \xrightarrow{LT} x'^\mu = \Lambda^\mu{}_\nu x^\nu,\ \partial_\mu = \Lambda^\nu{}_\mu\partial'_\nu,\ \Psi(x) \xrightarrow{LT} \Psi'(x') = S(\Lambda)\Psi(x) \tag{3.15}$$

The invariant of Dirac equation requires

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \mapsto (i\gamma^\mu\partial'_\mu - m)\Psi'(x') = 0 \tag{3.16}$$

$$\rightarrow (iS\gamma^\mu S^{-1}\Lambda^\nu{}_\mu\partial'_\nu - m)\Psi'(x') = 0 \tag{3.17}$$

$$\rightarrow S\gamma^\mu S^{-1}\Lambda^\nu{}_\mu = \gamma^\nu,\ \text{or}\ S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu \tag{3.18}$$

For infinitesimal Lorentz transforamtion, we can write

$$\Lambda^\mu{}_\nu(\omega) \simeq \delta^\mu{}_\nu + \omega^\mu{}_\nu + .. \mapsto S(\omega) \simeq 1 - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu} + ... \tag{3.19}$$

with $\omega^{\mu\nu} = -\omega^{\nu\mu}$. From (3.13), we will have

$$(\delta^\mu{}_\nu + \omega^\mu{}_\nu)\gamma^\nu = \left(1 + \frac{i}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}\right)\gamma^\mu\left(1 - \frac{i}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}\right) \tag{3.20}$$

$$\text{Since}:\ \omega^\mu{}_\nu\gamma^\nu = \frac{1}{2}\delta^\mu{}_\alpha\gamma_\beta\omega^{\alpha\beta} + \frac{1}{2}\delta^\mu{}_\beta\gamma_\alpha\omega^{\beta\alpha} = \frac{1}{2}\left(\delta^\mu{}_\alpha\gamma_\beta - \delta^\mu{}_\beta\gamma_\alpha\right)\omega^{\alpha\beta} \tag{3.21}$$

$$(3.15) \mapsto [\gamma^\mu, \sigma_{\alpha\beta}] = 2i(\delta^\mu{}_\alpha\gamma_\beta - \delta^\mu{}_\beta\gamma_\alpha) \rightarrow \sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta] \tag{3.22}$$

Next let us determine Lorentz transformations of Dirac spinors, and its Dirac conjugation:

$$\Psi(x) \xrightarrow{LT} \Psi'(x') = S\Psi(x) \tag{3.23}$$

$$\begin{aligned}\bar{\Psi}(x) &\xrightarrow{LT} \bar{\Psi}'(x') = \Psi^\dagger(x)S^\dagger\gamma^0 = \Psi^\dagger(x)\gamma^0\gamma^0S^\dagger\gamma^0 \\ &\mapsto \gamma^0S^\dagger\gamma^0 = S^{-1} \to \bar{\Psi} \xrightarrow{LT} \bar{\Psi}'(x') = \bar{\Psi}(x)S^{-1}\end{aligned} \tag{3.24}$$

Then we have

$$\bar{\Psi}(x)\Psi(x) \xrightarrow{LT} \bar{\Psi}'(x')\Psi'(x') = \bar{\Psi}(x)S^{-1}S\Psi(x) = \bar{\Psi}(x)\Psi(x) \tag{3.25}$$

$$\begin{aligned}\bar{\Psi}(x)\partial_\mu\Psi(x) \xrightarrow{LT} \bar{\Psi}'(x')\partial'_\mu\Psi'(x') &= \bar{\Psi}(x)S^{-1}\Lambda^\nu{}_\mu\partial_\nu S\Psi(x) \\ &= \Lambda^\nu{}_\mu\bar{\Psi}(x)\partial_\nu\Psi(x)\end{aligned} \tag{3.26}$$

3.1.3 Free field solutions

Since energy of Dirac spinor field is

$$E = \vec{\alpha}\cdot\vec{p} + \beta m \mapsto E^2 = \vec{p}^2 + m^2,\ E = E_p = \pm\sqrt{\vec{p}^2 + m^2}$$

It contains positive and negative energy parts. Dirac was interpreted the negative energy part as a hole in negative energy sea $E \leq -m^2$, in the *Dirac hole theory*. The positive and negative energy parts always created in pair from this negative energy sea or always destroyed (annihilated) from pair into this negative energy sea. So that we must determine solutions of Dirac equation for both positive and negative energy fields.

a) Positive energy solution: For $E > 0$, we use the trial solution in the form

$$\Psi(x) \sim U(k,s)e^{-ik\cdot x} \xrightarrow{DE} (\gamma^\mu k_\mu - m)U(k,s) = 0 \tag{3.27}$$

Rewrite Dirac spinor, let us denote $\sigma^0 = 1_{2x2} \equiv 1$, as

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \xrightarrow{DE} \begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & -E-m \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \tag{3.28}$$

$$\mapsto (E-m)u_1 - (\vec{\sigma}\cdot\vec{k})u_2 = 0 \tag{3.29}$$

$$(\vec{\sigma}\cdot\vec{k})u_1 - (E+m)u_2 = 0 \tag{3.30}$$

From (3.30), let us choose

$$u_1 = \chi_s \mapsto \sum_s \chi_s^\dagger\chi_s = 1,\ \sum_s \chi_s\chi_s^\dagger = 1_{2x2}\ (\textit{spinor basis}) \tag{3.31}$$

$$\text{and } u_2 = \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \mapsto U(k,s) = N\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \end{pmatrix} \tag{3.32}$$

Let us determine its normalization

$$U^\dagger = \begin{pmatrix} \chi_s^\dagger & \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s^\dagger \end{pmatrix}$$

$$\mapsto 2E = \sum_s U^\dagger U = |N|^2 \left(1 + \frac{(\vec{\sigma}\cdot\vec{k})^2}{(E+m)^2}\right) \sum_s \chi_s^\dagger \chi_s$$

$$= |N|^2 \left(1 + \frac{\vec{k}^2}{(E+m)^2}\right) = |N|^2 \left(\frac{2E}{E+m}\right) \to N = \sqrt{E+m} \tag{3.33}$$

After we have used Pauli matrix identity

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + \vec{a}\times\vec{b},$$

and have used the quadratic energy-momentum relation $E^2 = \vec{k}^2 + m^2$. We also have

$$\bar{U} = \begin{pmatrix} \chi_s^\dagger & -\frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s^\dagger \end{pmatrix}$$

$$\mapsto 2m = \sum_s \bar{U} U = |N|^2 \left(1 - \frac{(\vec{\sigma}\cdot\vec{k})^2}{(E+m)^2}\right) \sum_s \chi_s^\dagger \chi_s$$

$$= |N|^2 \left(1 - \frac{\vec{k}^2}{(E+m)^2}\right) = |N|^2 \left(\frac{2m}{E+m}\right) \mapsto N = \sqrt{E+m} \tag{3.34}$$

Let us determine its completeness relation

$$\sum_s U\bar{U} = (E+m) \sum_s \begin{pmatrix} \chi_1 \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \end{pmatrix} \begin{pmatrix} \chi_s^\dagger & -\frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s^\dagger \end{pmatrix}$$

$$= (E+m) \begin{pmatrix} 1 & -\frac{\vec{\sigma}\cdot\vec{k}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m} & -\frac{(\vec{\sigma}\cdot\vec{k})^2}{(E+m)^2} \end{pmatrix} \sum_s \chi_s \chi_s^\dagger$$

$$= \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & E-m \end{pmatrix} \equiv \gamma^\mu k_\mu + m = \not{k} + m \tag{3.35}$$

b) Negative energy solution: For $E < 0$, we use trial solution in the form

$$\Psi(x) \sim V(k,s) e^{ik\cdot x} \xrightarrow{DE} (\gamma^\mu k_\mu + m) V(k,s) = 0 \tag{3.36}$$

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & -E+m \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \tag{3.37}$$

$$\mapsto (E+m)v_1 - (\vec{\sigma}\cdot\vec{k})v_2 = 0 \tag{3.38}$$

$$(\vec{\sigma}\cdot\vec{k})v_1 - (E-m)v_2 = 0 \tag{3.39}$$

$$\text{We choos } v_2 = \chi_2(basis\ spinor),\ v_1 = \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \tag{3.40}$$

Then we have

$$V(k,s) = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \\ \chi_s \end{pmatrix} \tag{3.41}$$

Let us determine its normalization

$$\begin{aligned} \sum_s V^\dagger V &= |N|^2 \sum_s \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s^\dagger & \chi_s^\dagger \end{pmatrix} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \\ \chi_s \end{pmatrix} \\ = |N|^2 \left(\frac{(\vec{\sigma}\cdot\vec{k})^2}{(E+m)^2} + 1 \right) &\sum_s \chi_s^\dagger \chi_s = |N|^2 \left(\frac{\vec{k}^2}{(E+m)^2} + 1 \right) \\ &= |N|^2 \frac{2E}{E+m} \equiv 2E \mapsto N = \sqrt{E+m} \end{aligned} \tag{3.42}$$

We also have

$$\begin{aligned} \sum_s \bar{V} V &= |N|^2 \sum_s \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s^\dagger & -\chi_s^\dagger \end{pmatrix} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \\ \chi_s \end{pmatrix} \\ = |N|^2 \left(\frac{(\vec{\sigma}\cdot\vec{k})^2}{(E+m)^2} - 1 \right) &\sum_s \chi_s^\dagger \chi_s = |N|^2 \left(\frac{\vec{k}^2}{(E+m)^2} - 1 \right) \\ &= |N|^2 \frac{2m}{E+m} \equiv 2m \mapsto N = \sqrt{E+m} \end{aligned} \tag{3.43}$$

So that

$$V(k,s) = \sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \\ \chi_s \end{pmatrix} \tag{3.44}$$

Let us determine its completeness relation

$$\begin{aligned} \sum_s V(k,s)\bar{V}(k,s) &= (E+m) \sum_s \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \\ \chi_s \end{pmatrix} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s^\dagger & -\chi_s^\dagger \end{pmatrix} \\ &= (E+m) \begin{pmatrix} \frac{(\vec{\sigma}\cdot\vec{k})^2}{(E+m)^2} & -\frac{\vec{\sigma}\cdot\vec{k}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m} & -1 \end{pmatrix} \sum_s \chi_s \chi_s^\dagger \\ &= \begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & -E-m \end{pmatrix} = \gamma^\mu k_\mu - m \equiv \not{k} - m \end{aligned} \tag{3.45}$$

After we have used the fact that $\frac{\vec{k}^2}{E+m} = \frac{E^2-m^2}{E+m} = E - m$.

The general free field solution is written in term of Fourier integral with constraint condition of its energy dispersion $E_k = \sqrt{\vec{k}^2 + m^2}$ as

$$\begin{aligned} \Psi(x) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \sum_s &\left(a(k,s)U(k,s)e^{-ik\cdot x} + b^*(k,s)V(k,s)e^{ik\cdot x} \right) \\ &\times (2\pi)\delta(\omega^2 - E_k^2) \end{aligned} \tag{3.46}$$

Since

$$\delta(\omega^2 - E_k^2) = \frac{1}{2E_k}\left(\delta(\omega - E_k) + \delta(\omega + E_k)\right)$$

Then we have

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^3 2E - k} \sum_s \left(a(k,s)U(k,s)e^{-ik\cdot x} + b^*(k,s)V(k,s)e^{ik\cdot x} \right)_{\omega=E_k} \quad (3.47)$$

3.1.4 Dirac Hamiltonian

The conjugate momentum field of $\partial_0\Psi$ is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\partial_0\Psi} = i\bar{\Psi}\gamma^0 = i\Psi^\dagger \quad (3.48)$$

The Dirac Hamiltonian is then derived appear in the form

$$\mathcal{H} = \pi i\bar{\Psi}\gamma^0\partial_0\Psi - \mathcal{L} = -i\bar{\Psi}\vec{\gamma}\cdot\nabla\Psi + m\bar{\Psi}\Psi \quad (3.49)$$

$$\mapsto H = \int d^3x \left(-i\bar{\Psi}\vec{\gamma}\cdot\nabla\Psi + m\bar{\Psi}\Psi \right) \quad (3.50)$$

From above we have

$$\bar{\Psi}(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_s \Big(b(k,s)\bar{V}(k,s)e^{-ik\cdot x} + a^*(k,s)\bar{U}(k,s)e^{ik\cdot x} \Big)_{\omega=E_k} \quad (3.51)$$

$$\nabla\Psi(x) = i\int \frac{d^3k}{(2\pi)^3 2E_k}\vec{k} \sum_s \Big(a(k,s)U(k,s)e^{-ik\cdot x} - b^*(k,s)V(k,s)e^{ik\cdot x} \Big)_{\omega=E_k} \quad (3.52)$$

3.1.5 Spin, helicity and chirality spinors

The spin and helicity operators are defined as

$$\vec{S} = \frac{1}{2}\vec{\Sigma},\ \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix},\ \hat{h} = \vec{\Sigma}\cdot\hat{p} = \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} \quad (3.53)$$

where $\hat{p} = \frac{\vec{p}}{|\vec{p}|}$. We can observe that $[\vec{\Sigma}\cdot\hat{p}, H] = 0$, this means that helicity is conserved quantity but not Lorentz invariant since it is written in term of 3-vector. Further more we can observe that, with $H = \vec{\alpha}\cdot\vec{p} + \beta m$ for Dirac particle,

$$[\vec{\Sigma}, H] \neq 0,\ [\vec{L}, H] \neq 0,\ \text{but } [H, \vec{L} + \vec{S}] = 0,\ \text{where } \vec{L} = \vec{r}\times\vec{p} \quad (3.54)$$

$$\mapsto \vec{J} = \vec{L} + \vec{S} \text{ total angular momentum} \quad (3.55)$$

This shows that Dirac field particles are spin half fermions.

Let χ^h be the helicity spinor basis, i.e. $\hat{h}\chi^h = h\chi^h,\ h = \pm 1$, where $h = +1$ is called *right handedness* state and $h = -1$ is called *left handedness* state. The Dirac spinor can be expanded on the helicity spinor basis as

$$U(k,h) = N \begin{pmatrix} \chi^h \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi^h \end{pmatrix} \mapsto \hat{h}U(k,h) = hU(k,h) \tag{3.56}$$

$$V(k,h) = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi^h \\ \chi^h \end{pmatrix} \mapsto \hat{h}V(k,h) = hV(k,h) \tag{3.57}$$

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_h \left(a(k,h)U(k,h)e^{-ik\cdot x} + b^*(k,h)V(k,h)e^{ik\cdot x}\right)_{\omega=E_k} \tag{3.58}$$

Note that the spin spinor basis χ^s differs from the helicity spinor basis χ^h.

The chiral operator is defined in the form

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.59}$$

$$\mapsto \{\gamma_5, \gamma^\mu\} = 0,\ [\gamma_5, H] = 2m \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.60}$$

$$P_L = \frac{1}{2}(1-\gamma_5) = \frac{1}{2}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \tag{3.61}$$

$$P_R = \frac{1}{2}(1+\gamma_5) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.62}$$

Let us determine

$$\Psi^{(+)}_{k,h}(x) = N \begin{pmatrix} \chi_h \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_h \end{pmatrix} e^{-ik\cdot x} \tag{3.63}$$

$$\begin{aligned} \mapsto \Psi^{(+)}_{k,h,L}(x) = P_L \Psi^{(+)}_{k,h}(x) &= \frac{N}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \chi_h \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_h \end{pmatrix} e^{-ik\cdot x} \\ &= \frac{N}{2}\begin{pmatrix} +1 \\ -1 \end{pmatrix}\left(1 - \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\right)\chi_h e^{-ik\cdot x} \end{aligned} \tag{3.64}$$

Using the identity

$$1 - \frac{\vec{\sigma}\cdot\vec{k}}{E+m} = \frac{1}{2}\left(1 - \frac{|\vec{k}|}{E+m}\right)(1+\vec{\sigma}\cdot\hat{k}) + \frac{1}{2}\left(1 + \frac{|\vec{k}|}{E+m}\right)(1-\vec{\sigma}\cdot\hat{k})$$

$$1 - \frac{|\vec{k}|}{E+m} = 1 - \frac{\sqrt{E^2-m^2}}{E+m} = 1 - \frac{\sqrt{1-m^2/m^2}}{1+m/E}$$

$$\sim 1 - \left(1 - \frac{1}{2}\frac{m^2}{E^2} + ...\right)\left(1 - \frac{m}{E} + ...\right) \sim \frac{m}{E}\ in\ O(m/E)$$

$$1+\frac{|\vec{k}|}{E+m}=1+\frac{\sqrt{E^2-m^2}}{E+m}=1+\frac{\sqrt{1+m^2/E^2}}{1+m/E}$$
$$\sim 1+\left(1+\frac{1}{2}\frac{m^2}{E^2}+...\right)\left(1-\frac{m}{E}+...\right)\sim 2-\frac{m}{E}\ in\ O(m/E)$$

From (3.64), we have

$$\begin{aligned} P_L\Psi^{(+)}_{k,h,L}(x) &= \frac{N}{2}\begin{pmatrix} +1 \\ -1 \end{pmatrix}\left[\frac{m}{2E}(1+\vec{\sigma}\cdot\hat{k})\right. \\ &\left.+\left(1-\frac{m}{2E}\right)(1-\vec{\sigma}\cdot\vec{k})\right]\chi_h e^{-ik\cdot x} \end{aligned} \tag{3.65}$$

We observe that the left-chiral positive energy Dirac spinor contains the mixed right-handedness and left-handedness helicity states. The same analysis can be applied to $\Psi^{(+)}_{k,h,R}(x)$ and $\Psi^{(-)}_{k,h,R/L}(x)$.

3.1.6 Discrete symmetries of Dirac spinor

Discrete symmetries are *spatial inversion or parity* P, *time reversal* T, and *charge conjugation* C.

a) Parity transformation P: Let us determine the parity transformation

$$x \xrightarrow{P} x'=(t,-\vec{x}),\ \partial_\mu \xrightarrow{P} \partial'_\mu=(\partial_0,-\nabla) \tag{3.66}$$
$$p^\mu \xrightarrow{P} p'^\mu=(p^0,-\vec{p}),\ \Psi(x)\xrightarrow{P}\Psi'(x')=\Psi'(t,-\vec{x})=D(P)\Psi(x) \tag{3.67}$$

From Dirac equation

$$DE:\ (i\gamma^0\partial_0+i\vec{\gamma}\cdot\nabla-m)\Psi(t,\vec{x})=0$$

We will have

$$DE \xrightarrow{P} (i\gamma^0\partial_0-i\vec{\gamma}\cdot\nabla-m)\Psi'(t,-\vec{x})=0 \tag{3.68}$$

Since $\{\gamma^0,\gamma^i\}=0$, multiply through with γ^0 we will have

$$(i\gamma^0\partial_0+i\vec{\gamma}\cdot\nabla-m)\gamma^0\Psi'(t,-\vec{x})=0 \tag{3.69}$$

Thus we can get invariant Dirac equation under parity transformation if we have a transformation of Dirac spinor by a phase factor and multiplication with gamma matrix as

$$\Psi(t,\vec{x})\xrightarrow{P}\Psi'(x')=\gamma^0\Psi'(t,-\vec{x})\equiv\Psi(t,\vec{x}) \tag{3.70}$$
$$\gamma^0\gamma^0=1\rightarrow\Psi'(x')=\gamma^0\Psi(x)\mapsto D(P)=\gamma^0 \tag{3.71}$$

with arbitrary phase factor. Let us determine Lorentz transformation (LT) and parity transformation (P) of products of Dirac spinors

$$\bar{\Psi}(x)\Psi(x) \xrightarrow{LT} \bar{\Psi}'(x')\Psi'(x') = \Psi^\dagger(x)S^\dagger\gamma^0 S\Psi(x)$$
$$= \Psi^\dagger(x)\gamma^0 \underbrace{\gamma^0 S^\dagger \gamma^0}_{=S^{-1}} S\Psi(x) = \Psi^\dagger(x)\gamma^0\Psi(x) = \bar{\Psi}(x)\Psi(x) \quad (3.72)$$

$$\bar{\Psi}(x) \xrightarrow{P} \bar{\Psi}'(x')\Psi'(x') = \Psi'^\dagger(x')\gamma^0\Psi'(x') = (\gamma^0\Psi(x))^\dagger \underbrace{\gamma^0\gamma^0}_{=1} \Psi(x)$$
$$= \Psi^\dagger(x)\gamma^0\Psi(x) = \bar{\Psi}(x)\Psi(x) \quad (3.73)$$

This means that $\bar{\Psi}(x)\Psi(x)$ is *Lorentz scalar*. Next let us determine

$$J'^\mu(x') = \bar{\Psi}'(x')\gamma^\mu\Psi'(x') = \Psi'^\dagger(x')\gamma^0\gamma^\mu\Psi'(x')$$
$$= \Psi^\dagger(x)S^\dagger\gamma^0\gamma^\mu S\Psi(x) = \Psi^\dagger(x)\gamma^0 \underbrace{\gamma^0 S^\dagger\gamma^0}_{=S^{-1}} \gamma^\mu S\Psi(x)$$
$$= \Psi^\dagger\gamma^0 \underbrace{S^{-1}\gamma^\mu S}_{\Lambda^\mu{}_\nu\gamma^\nu} \Psi(x) = \Lambda^\mu{}_\nu\bar{\Psi}(x)\gamma^\nu\Psi(x) \equiv \Lambda^\mu{}_\nu J^\nu \quad (3.74)$$

$$PJ^\mu(x) = \Psi'^\dagger(x')\gamma^0\gamma^\mu\Psi'(x') = \Psi^\dagger(x)\gamma^0\gamma^0\gamma^\mu\gamma^0\Psi(x)$$
$$= \bar{\Psi}(x)\gamma^0\gamma^\mu\gamma^0\Psi(x) = \bar{\Psi}(x)\gamma'^\mu\Psi(x),\ \gamma'^\mu = (\gamma^0, -\vec{\gamma}) \quad (3.75)$$
$$J^0 \xrightarrow{P} J^0,\ \vec{J} \xrightarrow{P} -\vec{J} \quad (3.76)$$

We thus call that J^μ is an *polar vector*, i.e., its spatial component changed sign under parity. In general we will have

Quantity	Classification	Parity
$\bar{\Psi}\Psi$	Scalar (S)	+
$\bar{\Psi}\gamma^5\Psi$	Pseudoscalar (P)	-
$\bar{\Psi}\gamma^\mu\Psi$	Polar vector (V)	+
$\bar{\Psi}\gamma^\mu\gamma^5\Psi$	Axial vector (A)	-
$[\bar{\Psi}[\gamma^\mu,\gamma^\nu]\Psi$	Tensor (T)	+

b) Time reversal transformation T: Let us determine the time reversal transformation, dealing with complex conjugation (cc) of the plane wave solution, in the forms

$$x^\mu \xrightarrow{T} x'^\mu = (-t, \vec{x}),\ \partial_\mu \xrightarrow{T} \partial'_\mu = (-\partial_0, \nabla) \quad (3.77)$$
$$p^\mu \xrightarrow{T} p'^\mu = (p^0, -\vec{p}),\ \Psi(x) \xrightarrow{T} \Psi'(x') = D(T)\Psi^*(x) \quad (3.78)$$

From Dirac equation (DE)

$$DE \xrightarrow{T} (-i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\Psi'(x') = 0 \qquad (3.79)$$

$$\xrightarrow{c.c.} (i\gamma^0\partial_0 - i\gamma'^i\partial_i - m)\Psi'^*(x') = 0,\ \partial'^i = (\gamma^1, -\gamma^2, \gamma^3) \qquad (3.80)$$

$$\xrightarrow{-i\gamma^1\gamma^3\times} (i\gamma^0\partial_0 - i\gamma'^i\partial_i - m)(-i)\gamma^1\gamma^3\Psi'^*(x') - 2\delta^{1i}\partial'_i\gamma^3\Psi'^*(x') + 2\delta^{3i}\partial'_i\gamma^1\Psi'^*(x') = 0 \qquad (3.81)$$

$$\mapsto (i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\underbrace{(-i)\gamma^1\gamma^3\Psi'^*(x)}_{=\Psi(x)} = 0,\ i = 1,2,3 \qquad (3.82)$$

$$\mapsto \Psi'(x') = i\gamma^1\gamma^3\Psi^*(x) = D(T)\Psi(x),\ D(T) = i\gamma^1\gamma^3 \qquad (3.83)$$

After we have inserted the identity $\gamma^1\gamma^1 = 1 = \gamma^3\gamma^3$ into (3.81), and also used the fact that $\gamma^3\gamma^1 = -\gamma^1\gamma^3$.

c) Charge conjugation transformation C: Let us determine the charge conjugation, deal only with the change from particle into anti-particle or vice versa of plane wave solution (i.e.this corresponds to Dirac conjugation (dc) of Ψ), in the form

$$\Psi(x) \xrightarrow{C} \Psi'(x) = D(C)\bar{\Psi}^T(x) \text{ or } \Psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix} \xrightarrow{C} \bar{\Psi}^T = \begin{pmatrix} \chi^\dagger \\ -\eta^\dagger \end{pmatrix} \qquad (3.84)$$

From Dirac equation

$$\xrightarrow{Transposition(T)+dc} (i(\gamma^\mu)^T\partial_\mu + m)\bar{\Psi}^T(x) = 0 \qquad (3.85)$$

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^\mu \mapsto (\gamma^\mu)^T = \gamma^0\gamma^{*\mu}\gamma^0,\ \gamma^{*\mu} = (\gamma^0, \gamma^1, -\gamma^2, \gamma^3)$$
$$\mapsto \gamma^0(i\gamma^{*\mu}\partial_\mu + m)\gamma^0\bar{\Psi}^T(x) = 0 \qquad (3.86)$$

$$\xrightarrow{-i\gamma^2\times} \gamma^0(-\gamma^{*\mu}\partial_\mu - im)\gamma^2\gamma^0\bar{\Psi}^T(x) - 2\delta^{2\mu}\partial_\mu\gamma^0\bar{\Psi}^T(x) = 0$$
$$\mapsto i\gamma^0(i\gamma^\mu\partial_\mu - m)i\gamma^2\gamma^0\bar{\Psi}^T(x) = 0 \qquad (3.87)$$

$$i\gamma^2\gamma^0\bar{\Psi}^T(x) \xleftarrow{C} \Psi(x) \mapsto D(C) = i\gamma^2\gamma^0 \qquad (3.88)$$

d) The PCT theorem: Let us determine

$$\Psi(x) \xrightarrow{PCT} \Psi'(x') \equiv D(P)D(C)D(T)\Psi(x) \qquad (3.89)$$

$$D(P)D(C)D(T) = \gamma^0 i\gamma^2\gamma^0 i\gamma^1\gamma^3 = i\gamma^0\gamma^5 \qquad (3.90)$$

$$\{\gamma^5, \gamma^\mu\} = 0 \qquad (3.91)$$

This means that PCT is another symmetry of Dirac field theory.