# 4 Weyl and Majorana Spinor Fields

## 4.1 Weyl spinor

### 4.1.1 Weyl representation of gamma matrix

Let us determine massless Dirac equation

$$i\gamma^\mu \partial_\mu \Psi(x) = 0 \tag{4.1}$$

where $\gamma^\mu$ is Dirac gamma matrix, satisfy Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Solution of (4.1) cannot derived within Dirac representation of the gamma matrices as

$$\gamma^0 = \beta = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \ \gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \ i = 1, 2, 3$$

where $\sigma^0 = 1$ is 2x2 identity and $\{\sigma^i\}$ is a set of Pauli matrices. On the other hand, solution of (4.1) can be well derive within Weyl representation of the gamma matrix as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \text{ where } \sigma^\mu = (1, \vec{\sigma}), \ \bar{\sigma}^\mu = (1, -\vec{\sigma}) \tag{4.2}$$

$$\mapsto \{\sigma^\mu, \bar{\sigma}^\nu\} = 2g^{\mu\nu}, \ \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

Trial positive energy solution of (4.1) is written as $\Psi^{(+)}(x) \sim U(k)e^{-ik\cdot x}$, so that from (4.1) we will have

$$\gamma^\mu k_\mu U(k, h) = 0 \tag{4.4}$$

$$\text{For } U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \sigma^\mu k_\mu \\ \bar{\sigma}^\mu k_\mu & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \tag{4.5}$$

$$\mapsto 0 = \bar{\sigma}^\mu k_\mu u_1 = (k^0 + \vec{\sigma}\cdot\vec{k})u_1 = k^0(1 + \hat{h})u_1 \to \hat{h}u_1 = -u_1 \tag{4.6}$$

$$\mapsto 0 = \sigma^\mu k_\mu u_2 = (k^0 - \vec{\sigma}\cdot\vec{k})u_2 = k^0(1 - \hat{h})u_2 \to \hat{h}u_2 = +u_2 \tag{4.7}$$

after we have used the fact that $k^2 = 0 \mapsto (k^0)^2 = |\vec{k}|^2$, and we have defined *helicity operator* $\hat{h} = \vec{\sigma}\cdot\hat{k}$, where $\hat{k} = \vec{k}/|\vec{k}|$. Note also that

$$\hat{h}\chi_h = h\chi_h, \ h = \pm 1$$

where $h = +1$ is called *right handedness helicity* and $h = -1$ is called *left handedness helicity*, and $\chi_h$ is called *helicity-spinor*. From above we observe that $u_1$ is left-handedness helicity spinor, and $u_2$ is right handedness helicity spinor. Let us denote them as

$$u_1(k) = \chi_{-1}(k) = \eta(k), \ u_2(k) = \chi_+(k) = \bar{\psi}(k) \mapsto U(k) = \begin{pmatrix} \eta(k) \\ \bar{\psi}(k) \end{pmatrix} \tag{4.8}$$

$$\eta(x) \sim \eta(k)e^{-ik\cdot x}, \bar{\psi}(x) \sim \bar{\psi}(k)e^{-ik\cdot x} \mapsto \Psi^{(+)}(x) = \begin{pmatrix} \eta(x) \\ \bar{\psi}(x) \end{pmatrix} \tag{4.9}$$

For the trial negative energy solution, we write $\Psi^{(-)}(x) \sim V(k)e^{ik\cdot x}$, from (4.1) we have

$$-\gamma^\mu k_\mu V(k) = 0 \tag{4.10}$$

This shows that $V(k)$ satisfy the same equation as $U(k)$ but must differ from $U(k)$. From (4.8), we have only one way to write for the different solution

$$V(k) = \begin{pmatrix} v_1(k) \\ v_2(k) \end{pmatrix}, \mapsto v_1 = u_1,\ v_2 = -u_2, \mapsto V(k) = \begin{pmatrix} \eta(k) \\ -\bar{\psi}(k) \end{pmatrix} \tag{4.11}$$

$$\mapsto \Psi^{(-)}(x) = \begin{pmatrix} \eta(x) \\ -\bar{\psi}(x) \end{pmatrix} \tag{4.12}$$

This means that Weyl solution of massless Dirac equation is well written in terms of helicity spinors.

### 4.1.2 Chirality of massless Dirac spinor

Chiral operators are defined as

$$P_{R/L} = \frac{1}{2}(1 \pm \gamma^5) \mapsto P_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ P_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.13}$$

Chirality of Dirac spinor is then appear as

$$\Psi_R^{(\pm)}(x) = P_R \Psi^{(\pm)}(x) = \begin{pmatrix} 0 \\ \pm\bar{\psi}(x) \end{pmatrix} \tag{4.14}$$

$$\Psi_L^{(\pm)}(x) = P_L \Psi^{(\pm)}(x) = \begin{pmatrix} \eta(x) \\ 0 \end{pmatrix} \tag{4.15}$$

This shows that right chiral Dirac spinor is connected to right handedness helicity spinor, and left chiral Dirac spinor is connected to right handedness helicity spinor

### 4.1.3 Dirac Lagrangian in terms of Weyl spinors

The massless Dirac Lagrangian is

$$\begin{aligned}\mathcal{L} = \bar{\Psi} i\gamma^\mu \partial_\mu \Psi &= i \begin{pmatrix} \bar{\psi}^\dagger & \eta^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \partial_\mu \begin{pmatrix} \eta \\ \bar{\psi} \end{pmatrix} \\ &= i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta + i\bar{\psi}^\dagger \sigma^\mu \partial_\mu \bar{\psi} \end{aligned} \tag{4.16}$$

This leads to Weyl equations for left-handedness and right handedness spinors as

$$i\bar{\sigma}^\mu \partial_\mu \eta = 0, \text{ and } i\sigma^\mu \partial_\mu \bar{\psi} = 0 \tag{4.17}$$

### 4.1.4 Spinor indices

Let us assign indices for left handedness and right handedness spinors in the form

$$\eta \mapsto \eta^{\alpha},\ \eta^{\dagger} \mapsto (\eta^{\alpha})^{\dagger} = \bar{\eta}^{\dot{\alpha}} \tag{4.18}$$

$$\bar{\psi} \mapsto \bar{\psi}_{\dot{\alpha}},\ \bar{\psi}^{\dagger} \mapsto (\bar{\psi}_{\dot{\alpha}})^{\dagger} = \psi_{\alpha} \tag{4.19}$$

with $\alpha, \beta, ... = 1, 2$ and $\dot{\alpha}, \dot{\beta}, ... = \dot{1}, \dot{2}$. We can lift or lower these indices by using total antisymmetric tensor as

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}} \tag{4.20}$$

$$\rightarrow \eta_{\alpha} = \epsilon_{\alpha\beta}\eta^{\beta}, \eta^{\alpha} = \epsilon^{\alpha\beta}\eta_{\beta},\ \text{and}\ \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}},\ \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}} \tag{4.21}$$

The contractions are defined as

$$\eta \cdot \eta = \eta_{\alpha}\eta^{\alpha},\ \text{and}\ \bar{\psi} \cdot \bar{\psi} = \bar{\psi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} \tag{4.22}$$

Similarly we also assign the spinor indices to gamma matrix in the form

$$i\eta^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\eta \mapsto i\bar{\eta}^{\dot{\alpha}}\bar{\sigma}^{\mu}_{\dot{\alpha}\beta}\partial_{\mu}\eta^{\beta} \tag{4.23}$$

$$i\bar{\psi}^{\dagger}\sigma^{\mu}\partial_{\mu}\bar{\psi} \mapsto i\psi_{\alpha}\sigma^{\mu,\alpha\dot{\beta}}\partial_{\mu}\bar{\psi}_{\dot{\beta}} \tag{4.24}$$

$$\{\sigma^{\mu}, \bar{\sigma}^{\nu}\} = \gamma^{\mu}\bar{\sigma}^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu} \mapsto \sigma^{\mu,\alpha\dot{\beta}}\bar{\sigma}_{\dot{\beta}\alpha} + \bar{\sigma}^{\nu}_{\dot{\alpha}\beta}\sigma^{\mu,\beta\dot{\alpha}} = 2g^{\mu\nu} \tag{4.25}$$

$$\sigma^{\mu}k_{\mu} \mapsto \sigma^{\mu,\alpha\dot{\beta}}k_{\mu} \equiv k^{\alpha\dot{\beta}} = \begin{pmatrix} k^0 - k^3 & -k^1 + ik^2 \\ -k^1 - ik^2 & k^0 + k^3 \end{pmatrix} \tag{4.26}$$

$$\bar{\sigma}^{\mu}k_{\mu} = \bar{\sigma}^{\mu}_{\dot{\alpha}\beta}k_{\mu} \equiv k_{\dot{\alpha}\beta} = \begin{pmatrix} k^0 + k^3 & k^1 - ik^2 \\ k^1 + ik^2 & k^0 - k^3 \end{pmatrix} \tag{4.27}$$

$$\det(k) = (k^0)^2 - |\vec{k}|^2 = k^2 = 0 \mapsto k^{\mu} - \textit{null vector} \tag{4.28}$$

### 4.1.5 Spinor repsentation of Lorentz transformation

Let us determine Lorentz transformation of Weyl equation (WE)

$$x \xrightarrow{LT} x' = \Lambda x,\ \eta(x) \xrightarrow{LT} \eta'(x') = S(\Lambda)\eta(x) \tag{4.29}$$

$$WE \xrightarrow{LT} iD\bar{\sigma}^{\mu}\partial_{\mu}\eta(x) = i\underbrace{D\bar{\sigma}^{\mu}D^{-1}}_{=\Lambda^{\mu}{}_{\nu}\bar{\sigma}^{\nu}}\partial_{\mu}D\eta(x) = i\bar{\sigma}^{\nu}\partial'_{\nu}\eta'(x') = 0 \tag{4.30}$$

From infinitesimal transformation $\Lambda \simeq 1 + \omega$, we have

$$D(\omega) = e^{-i\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}},\ \text{where}\ \ S^{\mu\nu} = \frac{1}{4}[\gamma^{\mu}, \gamma^{\nu}] \tag{4.31}$$

Since

$$[\gamma^{\mu}, \gamma^{\nu}] = \gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}$$

$$= \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix} = \begin{pmatrix} 4\sigma^{\mu\nu} & 0 \\ 0 & 4\bar{\sigma}^{\mu\nu} \end{pmatrix}$$

where

$$\sigma^{\mu\nu} = (\sigma^{\mu\nu})_\alpha{}^\beta, \ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}, \text{ and } S^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}$$

$$D(\omega) = \begin{pmatrix} M & 0 \\ 0 & \bar{M} \end{pmatrix}, \ M = e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}, \ \bar{M} = e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}}$$

## 4.2 Majorana spinor field

Majorana representation of Dirac gamma matrices are

$$\tilde{\gamma}^0 = \sigma^2 \otimes \sigma^1, \ \tilde{\gamma}^1 = i\sigma^1 \otimes 1, \ \tilde{\gamma}^2 = i\sigma^3 \otimes 1, \ \tilde{\gamma}^3 = i\sigma^2 \otimes \sigma^2 \quad (4.32)$$

Dirac equation will appear in the form

$$(i\tilde{\gamma}^\mu \partial_\mu - m)\tilde{\Psi}(x) = 0 \quad (4.33)$$

where $\tilde{\Psi}(x)$ is Majorana spinor. Since $\tilde{\gamma}^\mu$ is pure complex, this will make Dirac equation to be real and Majorana solution will be real spinor. For massless field, we can write

$$\tilde{\Psi}(x) = \begin{pmatrix} \eta_\alpha(x) \\ \bar{\eta}^{\dot{\alpha}}(x) \end{pmatrix} \mapsto (\tilde{\Psi}^\dagger)^T(x) = \tilde{\Psi}(x) \quad (4.34)$$