# 5 Classical Aspects of Vector Fields

## 5.1 Maxwell vector field

### 5.1.1 Maxwell field Lagrangian

Let us determine electromagnetic field as classical massless vector field, known as Maxwell field. Its dynamics is determined from Lorentz vector field $A^\mu(x)$, known as 4-potential, and its Lagrangian is written in term of field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,\ F^{\mu\nu} = -F^{\nu\mu} \mapsto F^{00} = F^{ii} = 0; i = 1,2,3 \tag{5.1}$$

$$F^{0i} = \partial_0 A^i + \partial_i A^0 = -E^i, \tag{5.2}$$

$$F^{ij} = -(\partial_i A^j - \partial_j A^i) = -\epsilon^{ijk} B^k \tag{5.3}$$

$$F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} \mapsto F_{0i} = -F^{0i},\ F_{ij} = F^{ij} \tag{5.4}$$

$$\text{Bianchi identity } \partial_\nu F_{\rho\sigma} + \partial_\sigma F_{\nu\rho} + \partial_\rho F_{\sigma\nu} = 0 \tag{5.5}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} F_{0i} F^{0i} - \frac{1}{2} F_{ij} F^{ij} = \frac{1}{2} E^i E^i - \frac{1}{2} B^k B^k \tag{5.6}$$

$$\mapsto L = -\frac{1}{4} \int d^3x F_{\mu\nu} F^{\mu\nu} \equiv \frac{1}{2} \int d^3x (\vec{E}^2 - \vec{B}^2) \tag{5.7}$$

$$S[A^\mu] = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} \tag{5.8}$$

### 5.1.2 Gauge symmetry

Let us determine a transformation of vector field

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi \mapsto F^{\mu\nu} \to F'^{\mu\nu} = F^{\mu\nu} \tag{5.9}$$

for arbitrary scalar function $\chi(x)$. This symmetry had got the name of *gauge symmetry* by Herman Weyl. To get the physical vector field, we have to break this symmetry by applying with *gauge fixing condition*, in which the commonly used conditions are

- Lorentz condition: $\partial_\mu A^\mu(x) = 0$,
- Coulomb condition: $\nabla \cdot \vec{A} = 0$.

With Lorentz gauge condition, its Euler-Lagrange equation becomes

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial_\mu A_\nu} = 0,\ \mathcal{L} = -\frac{1}{2} \partial_\mu A_\nu F^{\mu\nu} \tag{5.10}$$

$$\mapsto \partial_\mu F^{\mu\nu} = \partial^2 A_\nu - \partial_\nu \partial_\mu A^\mu \mapsto \partial^2 A^\mu = 0 \tag{5.11}$$

### 5.1.3 Free field solution

Its trial free field solution is

$$A^\mu(x) \sim \epsilon^\mu(k,\lambda)a(k,\lambda)e^{-ik\cdot x} \mapsto -k^2\epsilon^\mu(k,\lambda)a(k,\lambda) = 0 \tag{5.12}$$

$$\to k^2 = \omega^2 - \vec{k}\cdot\vec{k} = 0 \mapsto \omega^2 - \omega_k^2 = 0,\ \omega_k = |\vec{k}| \tag{5.13}$$

Its general free field solution is then written in term of Fourier expansion, with constrain from its dispersion, as

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\sum_\lambda \left[\epsilon^\mu(k,\lambda)a(k,\lambda)e^{-ik\cdot x}\right.$$
$$\left. + \epsilon^{*\mu}(k,\lambda)a^*(x)e^{ik\cdot x}\right](2\pi)\delta(\omega^2-\omega_k^2)\theta(\omega) \tag{5.14}$$

Using identity

$$\delta(f(x)) = \sum_i \frac{\delta(x-a_i)}{|f'(a_i)|},\ \text{where } f(a_i) = 0$$

$$\mapsto \delta(\omega^2-\omega_k^2) = \frac{1}{2\omega_k}[\delta(\omega-\omega_k)+\delta(\omega+\omega_k)]$$

Note that $\epsilon^\mu(k,\lambda)$ is a generic polarization tensor, satisfy orthogonality condition

$$\epsilon^{*\mu}(k,\lambda)\epsilon^\nu(k,\lambda') = g^{\mu\nu}\delta_{\lambda\lambda'} \tag{5.15}$$

### 5.1.4 Maxwell field Hamiltonian

The conjugate momentum field is derived in the form

$$\pi^\mu(x) = \frac{\partial\mathcal{L}}{\partial\partial_0 A_\mu} = -F^{0\mu}, \text{since } F^{00} = 0 \mapsto \pi^0 = 0 \tag{5.16}$$

This means that $A^0(x)$ is not a dynamical field, so that we will set $A^0 = 0$ for convenient. After applying Legendre transformation of the Lagrangian, we will get the vector field Hamiltonian in the form

$$\mathcal{H} = \pi^i\partial_0 A_i - \mathcal{L} \equiv -F^{0i}F_{0i} + \frac{1}{2}F^{0i}F_{0i} + \frac{1}{2}F^{ij}F_{ij}$$
$$= \frac{1}{2}F^{0i}F^{0i} + \frac{1}{2}F^{ij}F^{ij} = \frac{1}{2}(E^iE^i + B^kB^k) \tag{5.17}$$

$$\mapsto H = \frac{1}{2}\int d^3x\left(\vec{E}\cdot\vec{E}+\vec{B}\cdot\vec{B}\right) \tag{5.18}$$

### 5.1.5 Maxwell field with Coulomb condition

We can write $A^\mu(x) = (0,\vec{A}(x))$, and from free field solution we will have

$$\epsilon^\mu(k,\lambda) = (0,\hat{\epsilon}(k,\lambda)),\ \text{and } \vec{k}\cdot\hat{\epsilon} = 0 \tag{5.19}$$

$$\text{For } k^\mu = (\omega_k,0,0,k) \mapsto \hat{\epsilon}(k,\lambda) = (0,\epsilon^1,\epsilon^2,0) \to \lambda = 1,2 \tag{5.20}$$

That is $\vec{A}(x)$ is a transverse vector field, corresponds to transverse electric field $\vec{E} = -\partial_0\vec{A}$ and transverse magnetic field $\vec{B}(x) = \nabla\times\vec{A}(x)$.

### 5.1.6 Fermi's trick and Feynman gauge

From Lagrangian (5.6), its Euler-Lagrange equation without any gauge fixing condition reads

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu(x) = 0 \tag{5.21}$$

Note that the differential operator $D_{\mu\nu} = (g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)$ does not have its inverse, we can determine from its Fourier transformation as

$$-k^2\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)a^\nu(k) = 0 \tag{5.22}$$

This problem can be cured by using Fermi's trick, by adding to the Lagrangian (5.6) with gauge fixing condition as

$$(5.6 \mapsto \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial_\mu A^\mu)^2,\ \xi - \text{gauge parameter} \tag{5.23}$$

$$\begin{aligned} S[A^\mu] &= \int d^4x\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial_\mu A^\mu)^2\right\} \\ &= \int d^4x\left\{-\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{\xi}{2}(\partial_\mu A^\mu)^2\right\} \\ &= \frac{1}{2}\int d^4x\left\{A_\mu(g^{\mu\nu}\partial^2 - (1-\xi)\partial^\mu\partial^\nu)A_\nu\right\} \end{aligned} \tag{5.24}$$

$$\mapsto \mathcal{L} = \frac{1}{2}A^\mu(g_{\mu\nu}\partial^2 - (1-\xi)\partial_\mu\partial_\nu)A_\nu \tag{5.25}$$

$$EOM \mapsto (g_{\mu\nu}\partial^2 - (1-\xi)\partial_\mu\partial_\nu)A_\nu = 0 \tag{5.26}$$

$$\text{Plane wave solution} \mapsto -k^2\underbrace{\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)}_{=D_{\mu\nu}(k)}a^\mu(k) = 0 \tag{5.27}$$

$$\mapsto D^{-1}_{\mu\nu}(k) = \left(g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\frac{k^\mu k^\nu}{k^2}\right) \tag{5.28}$$

Feynman gauge condition is $\xi = 1$.

### 5.1.7 Energy-momentum tensor

From the Lagrangian (5.5), we can derive its energy-momentum tensor in the form

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial_\mu A_\rho}\partial^\nu A^\rho - g^{\mu\nu}\mathcal{L} = -F^{\mu\rho}\partial^\nu A_\rho + \frac{g^{\mu\nu}}{4}F^{\rho\sigma}F_{\rho\sigma} \tag{5.29}$$

$$\mapsto T^{00} = -F^{0i}F_{0i} + \frac{1}{2}F^{0i}F_{0i} + \frac{1}{2}F^{ij}F_{ij} = \frac{1}{2}(E^iE^i + B^kB^k) \equiv \mathcal{H} \tag{5.30}$$

$$\begin{aligned} T^{0i} &= -F^{0j}\partial^j A_k \equiv \frac{1}{2}E^j(\partial_i A^j - \partial_j A^i) = \frac{1}{2}E^j\epsilon^{ijk}B^k \\ &\mapsto \mathcal{P}^i = \frac{1}{2}\epsilon^{ijk}E^jB^k,\ \vec{P} = \frac{1}{2}\int d^3\vec{E}\times\vec{B} \end{aligned} \tag{5.31}$$

It is the Poynting vector.

### 5.1.8 Spin angular momentum tensor

From Lorentz transformation of a vector field is

$$x^\mu \xrightarrow{LT} x'^\mu = \Lambda^\mu{}_\nu x^\nu,\ A^\mu \xrightarrow{LT} A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x) \tag{5.32}$$

$$A'^\mu(x) = \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x) \mapsto \delta A^\mu(x) = \frac{1}{2}\omega^{\alpha\beta}\Lambda^{\mu\nu}M_{\alpha\beta}A_\nu(x)$$

$$\simeq \frac{1}{2}\omega^{\alpha\beta}\underbrace{g^{\mu\nu}M_{\alpha\beta}}_{=\Sigma^{\mu\nu}_{\alpha\beta}}A_\nu \tag{5.33}$$

The spin tensor of a vector field is

$$S^\mu{}_{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial\partial_\mu A^\rho}g^{\rho\sigma}M_{\alpha\beta}A_\sigma \mapsto S^0_{\alpha\beta} = -\frac{\partial\mathcal{L}}{\partial\partial_0 A^i}\delta^{ij}M_{\alpha\beta}A_j \tag{5.34}$$

$$\mapsto S^0_{ij} = L_{ij} = (\vec{E}\cdot\vec{A})M_{ij} \tag{5.35}$$

The next step of this analysis must be done on quantum level.

## 5.2 Electromagnetic duality

Let us determine the *dual field strength tensor*, which is defined in the form

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{5.36}$$

$$\mapsto \tilde{F}^{0i} = \frac{1}{2}\epsilon^{0ijk}F_{jk};\ i,j,k = 1,2,3$$

$$\tilde{F}^{01} = -B^1,\ \tilde{F}^{02} = -B^2,\ \tilde{F}^{03} = -B^3 \mapsto \vec{E} \xrightarrow{DT} \vec{B} \tag{5.37}$$

$$\mapsto \tilde{F}^{ij} = \frac{1}{2}\epsilon^{ij\mu\nu}F_{\mu\nu},\ \mu,\nu \neq i,j = 1,2,3$$

$$\tilde{F}^{12} = \frac{1}{2}\epsilon^{1230}F_{30} + \frac{1}{2}\epsilon^{1203}F_{03} = F_{03} = -F^{03} = E^3$$

$$\text{Similarly } \tilde{F}^{13} = -E^2,\ \tilde{F}^{23} = E^1 \mapsto \vec{B} \xrightarrow{DT} -\vec{E} \tag{5.38}$$

$$\mapsto \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -F_{\mu\nu}F^{\mu\nu} \tag{5.39}$$

In general, the dual symmetry of electromagnetic field is defined in the form

$$\vec{E} \xrightarrow{DT} \vec{E}\cos\theta + \vec{B}\sin\theta \tag{5.40}$$

$$\vec{B} \xrightarrow{DT} \vec{B}\cos\theta - \vec{E}\sin\theta \tag{5.41}$$

where $\theta$ is arbitrary scalar parameter. And the dual transformation of the source charge $\rho$ and current $\vec{j}$ are defined as

$$\rho_e \xrightarrow{DT} \rho_e \cos\theta + \rho_m \sin\theta \tag{5.42}$$

$$\rho_m \xrightarrow{DT} \rho_m \cos\theta - \rho_e \sin\theta \tag{5.43}$$

$$\vec{j}_e \xrightarrow{DT} \vec{j}_e \cos\theta + \vec{j}_m \sin\theta \tag{5.44}$$

$$\vec{j}_m \xrightarrow{DT} \vec{j}_m \cos\theta - \vec{j}_e \sin\theta \tag{5.45}$$

Note form (5.5), when apply with $\epsilon^{\mu\nu\rho\sigma}$, we observe that

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} + \epsilon^{\sigma\mu\nu\rho}\partial_\sigma F_{\nu\rho} + \epsilon^{\rho\sigma\mu\nu}\partial_\rho F_{\sigma\nu} = 0 \\ \mapsto \partial_\mu \frac{1}{2}\epsilon^{\mu\rho\sigma\nu} F_{\sigma\nu} \equiv \partial_\mu \tilde{F}^{\mu\rho} = 0 \end{aligned} \tag{5.46}$$

This shows that the dual field strength tensor also satisfy the Maxwell equation With a source term $\tilde{j}^\mu = (\rho_m, \vec{j}_m)$, where $\rho_m$ will be magnetic charge (monopole) density and Maxwell equation becomes

$$\partial_\mu \tilde{F}^{\mu\nu} = -\tilde{j}^\mu$$

This means that the magnetic charge should exist in the nature, according to this duality symmetry.

The dual symmetric Lagrangian can be written in the form

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu} + \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right) \tag{5.47}$$

And assume the existent of dual vector field

$$\tilde{F}^{\mu\nu} = \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu \tag{5.48}$$

Then we can introduce the complex vector field in the form

$$C^\mu = A^\mu + i\tilde{A}^\mu \mapsto D^{\mu\nu} = \partial^\mu C^\nu - \partial^\nu C^\mu \tag{5.49}$$

$$\mapsto \tilde{D}^{\mu\nu} = -iD^{\mu\nu} \text{ and } \mathcal{L} = -\frac{1}{8}D_{\mu\nu}D^{*\mu\nu} \tag{5.50}$$

$$EOM \to \partial_\mu D^{\mu\nu} = 0 \tag{5.51}$$

And the dual transformation can be written in term of simple U(1) gauge transformation as

$$C^\mu \xrightarrow{DT} e^{-i\theta}C^\mu \mapsto \begin{cases} A^\mu \xrightarrow{DT} A^\mu \cos\theta + \tilde{A}^\mu \sin\theta \\ \tilde{A}^\mu \xrightarrow{DT} \tilde{A}^\mu \cos\theta - A^\mu \sin\theta \end{cases} \tag{5.52}$$

$$\mapsto D^{\mu\nu} \xrightarrow{DT} e^{-i\theta}D^{\mu\nu} \tag{5.53}$$

And the Lagrangian (5.44) is invariant under duality transformation.

## 5.3 Born-Infeld Lagrangian

From Maxwell Lagrangian (5.6), it grows to infinity when $E \to \infty$ close to the point charge. To eliminate this infinity Born and Infeld put the upper value of the electric field at $b$ and proposed the new vector field Lagrangian of the form

$$
\begin{aligned}
\mathcal{L} &= b^2 \left(1 - \sqrt{1 - \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2}}\right) \\
&= b^2 \left(1 - \sqrt{1 - \frac{\vec{E}^2 - \vec{B}^2}{b^2}}\right) \xrightarrow[|\vec{E}|/b \ll 1]{} \frac{1}{2}(\vec{E}^2 - \vec{B}^2)
\end{aligned} \tag{5.54}
$$

in which we recover Maxwell Lagrangian in the limit of weak field $|\vec{E}|/b \ll 1$. It is similar to the case we have put the upper value of velocity $v$ within special relativity at the light speed $c$ and replace the free particle Lagrangian from

$$
L = mc^2 \left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) \xrightarrow[v/c \to 0]{} \frac{1}{2}mv^2
$$

A more general form of Born-Infeld Lagrangian is written in the form

$$
\begin{aligned}
\mathcal{L} &= b^2 \left(1 - \sqrt{1 - \det\left(g^{\mu\nu} + \frac{1}{b}F^{\mu\nu}\right)}\right) \\
&= b^2 \left(1 - \sqrt{1 + \frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16b^4}F_{\mu\nu}\tilde{F}^{\mu\nu}}\right)
\end{aligned} \tag{5.55}
$$