# 1 Functional Integral Representations

## 1.1 Path integral in quantum mechanics

### 1.1.1 Basic of quantum dynamics

Let $\psi(x,t)$ be a time-dependent state function of quantum system satisfy Schrodinger equation

$$i\partial_t\psi(x,t) = H\psi(x,t) \tag{1.1}$$

where $H = \frac{p^2}{2m} + V(x)$ is a system Hamiltonian. There will be a unitary time-operator $U(t,0)$ with the action

$$\psi(x,t) = U(t,0)\psi(x,0) \tag{1.2}$$

We can observe that the unitary operator itself satisfy the Schrodinger equation (1.1)

$$i\partial_t U(t,0) = HU(t,0) \to U(t,0) = e^{-iHt} \tag{1.3}$$

with the fact that $U(0,0) = 1$ and $H \neq H(t)$.

### 1.1.2 Feynman propagator

Using the normalization property of the state function $\int dx\psi^*(x,t)\psi(x,t) = 1$, one can simple write the identity

$$\psi(b,T) = \int da\psi(b,T)\psi^*(a,0)\psi(a,0) \equiv \int daK(b,a;T)\psi(a,0) \tag{1.4}$$

Mathematically $K(b,a;T)$ is known in the name of *integral kernel*, but physicists is known in the name of *Feynman propagator*, according to R. Feynman who first wrote this relation in his PhD thesis. The quantum meaning of Feynman propagator is *the probability amplitude finding a particle at* $a$ *at the initial time* 0 *and then finding it again at* $b$ *at later time* $T > 0$.

We can get more detail analysis of Feynman propagator, with the help of Dirac *bracket* notation, as in the following

$$\begin{aligned} K(b,a:T) &= \psi(b,T)\psi^*(a,0) =< b|\psi(T)>< \psi(0)|a> \\ &=< b|U(T,0)|\psi(0)>< \psi(0)|a> =< b|U(T,0)|a> \end{aligned} \tag{1.5}$$

This bring us to be able to do the calculation of the Feynman propagator. First we use the semi-group property of the unitary operator such that when we break the time interval $[T, 0]$ into N of small duration $\epsilon$ as

$$T = N\epsilon \to t_n = n\epsilon,\ \text{with } t_0 = 0,\ t_N = T \tag{1.6}$$

The we can write

$$U(T,0) = \underbrace{U(\epsilon)U(\epsilon)...U(\epsilon)}_{N-terms}, \quad U(\epsilon) = e^{-i\epsilon H} \tag{1.7}$$

From (1.5) we will have

$$\begin{aligned} K(b, a : T) &=< b|\underbrace{U(\epsilon)U(\epsilon)...U(\epsilon)}_{N-terms}|a> \\ = \Pi_{n=1}^{N-1}\left[\int dx_n\right] &< x_N|U(\epsilon)|x_{N-1} >< x_{N-1}|U(\epsilon)|x_{N-2} > ... \\ &... < x_n|U(\epsilon)|x_{n-1} > ... < x_2|U(\epsilon)|x_1 >< x_1|U(\epsilon)|x_0 > \qquad (1.8) \\ = \Pi_{n=1}^{N-1}\left[\int dx_n\right] &K(x_N, x_{N-1}; \epsilon)K(x_{N-1}, x_{N-2}; \epsilon)... \\ &...K(x_n, x_{n-1}; \epsilon)...K(x_2, x_1; \epsilon)K(x_1, x_0; \epsilon) \qquad (1.9) \end{aligned}$$

where we have used the expression

$$K(x_n, x_{n-1}; \epsilon) =< x_n|e^{-i\epsilon H}|x_{n-1} > . \tag{1.10}$$

In the limit of $N \to \infty$, or $\epsilon \to 0$, we will have

$$\begin{aligned} K(x_n, x_{n-1}; \epsilon) &=< x_n|e^{-i\epsilon H}|x_{n-1} >\simeq< x_n|e^{-ip^2/2m}e^{-i\epsilon V(x)}|x_{n-1} > \\ &\simeq e^{-i\epsilon V(x_n)} < x_n|e^{-i\epsilon p^2/2m}|x_{n-1} > \\ &\simeq e^{-i\epsilon V(x_n)} \int dp_n < x_n|e^{-i\epsilon p^2/2m}|p_n >< p_n|x_{n-1} > \\ &\simeq e^{-i\epsilon V(x_n)} \int dp_n e^{-i\epsilon p_n^2/2m} < x_n|p_n >< p_n|x_{n-1} > \\ &\simeq e^{-i\epsilon V(x_n)} \int \frac{dp_n}{2\pi} e^{-i\epsilon p_n^2/2m + ip_n(x_n - x_{n-1})} \\ &\simeq \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \exp\left(\frac{im\epsilon}{2}\left(\frac{x_n - x_{n-1}}{\epsilon}\right)^2 - i\epsilon V(x_n)\right) \qquad (1.11) \end{aligned}$$

after we have used the fact that $< x|p >= e^{ipx}/\sqrt{2\pi}$, and we have applied the Gaussian integration over $\iint dp_n$. Insertion into (1.9), we get

$$K(b,a;T) = \lim_{N\to\infty} \Pi_{n=1}^{N-1} \left[\int dx_n\right] \left(\frac{m}{2\pi i\epsilon}\right)^{N/1} \ldots$$
$$\times \exp\left(i\epsilon \sum_{n=1}^{N} \frac{m}{2}\left(\frac{x_n - x_{n-1}}{\epsilon}\right)^2 - V(x_n)\right\} \quad (1.12)$$
$$\to K(b,a;T) = \int \mathcal{D}[x] e^{iS[x]}, \quad S[x] = \int_0^T dtL \quad (1.13)$$
$$L = \frac{1}{2}m\dot{x}^2 - V(x) \quad (1.14)$$

It is known as *Feynman path integral* formula.

### 1.1.3 The generating function

According to its quantum meaning, we can do the quantum expectation value through the *generating function* which is written in term of path integral as

$$z[j] = \int \mathcal{D}[x] e^{iS[x]+i\int dtx(t)j(t)} \quad (1.15)$$

The we can have

$$< x >= \frac{\int \mathcal{D}[x]x(t)e^{iSx]}}{\int \mathcal{D}[x]e^{iS[x]}} = \frac{1}{z[j]}(-i)\frac{\partial z[j]}{\partial j(t)}\bigg|_{j=0} \quad (1.16)$$
$$< f(x) >= \frac{1}{z[j]} f\left((-i)\frac{\partial}{\partial j(t)}\right) z[j]\bigg|_{j-0} \quad (1.17)$$

## 1.2 Gaussian integrals

The method we have used is Gaussian integration, in this section we come to summarize of the integration.

### 1.2.1 Single-variable integration

$$I(a,b) = \int_{-\infty}^{+\infty} dx e^{-ax^2/2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/2a} \quad (1.18)$$

### 1.2.2 Multi-variable integration

$$
\begin{aligned}
I(\{a_i\},\{b_i\}) = \Pi_{i-1}^{N}\left[\int dx_i\right] e^{-\frac{1}{2}\sum_{i=1}^{N} a_i x_i^2 + b_i x_i} \\
= \frac{\pi^{N/2}}{\sqrt{\Pi_{i=1}^{N} a_i}} e^{\sum_{i=1}^{N} b_i^2/2a_i}
\end{aligned}
\tag{1.19}
$$

Define matrices $A = \{a_i\delta_{ij}\},\ b = \{b_i\},\ b = \{x_i\}$, we can rewrite (1.19) in matrix form as

$$
I(A,B) = \int dX e^{-\frac{1}{2}b\cdot A\cdot x + b\cdot x} = \pi^{N/2} det[A]^{-1/2} e^{\frac{1}{2}b\cdot A^{-1}\cdot b} \tag{1.20}
$$

$$
\to \pi^{-N/2}\Pi_{i=1}^{N}\left[\int dx_i\right] \equiv \int \mathcal{D}[x] \tag{1.21}
$$

$$
\int \mathcal{D}[x] e^{-\frac{1}{2}x\cdot A\cdot x + b\cdot x} = det[A]^{-1/2} e^{\frac{1}{2}b\cdot A^{-1}\cdot b} \tag{1.22}
$$

## 1.3 Functional integral of bosonic field

Let $\phi(x,t)$ be a bosonic field with the Lagrangian $\mathcal{L}_0 = \frac{1}{2}((\partial\phi)^2 - m^2\phi^2)$, we can extend the path integral to be the functional integral by doing the following substitutions

$$
x(t) \quad \to \phi(x,t) \tag{1.23}
$$

$$
z[j(t)] \quad \to Z_0[J(x,t)] = \int \mathcal{D}[\phi] e^{iS_0[\phi] + i\int dx \int dt \phi(x,t) J(x,t)} \tag{1.24}
$$

$$
S_0[\phi] = \int dx \int dt \mathcal{L}_0(\phi, \partial_\mu\phi) \tag{1.25}
$$

From now we come to use a shorthand notations of $(x,t) = x$ of dim-4, and

$$
\int d^4x \phi(x) J(x) = (\phi, J) \tag{1.26}
$$

$$
\begin{aligned}
S_0[\phi] = \int d^4x \mathcal{L} = \int d^4x \frac{1}{2}(\partial\phi\cdot\partial\phi - m^2\phi^2) \\
= -\frac{1}{2}\int d^2x \phi(\partial^2 + m^2)\phi = -\frac{1}{2}(\phi, G_0^{-1}\phi)
\end{aligned}
\tag{1.27}
$$

where we have ignore the total derivative term, and $G_0^{-1}$ is the differential operator appears in the free field equation of motion. We can apply the Gaussian integral to evaluate $Z_0[J]$ and results to

$$\int \mathcal{D}[\phi] = \lim_{N\to\infty} \pi^{-N/2}\Pi_{n=1}^{N}\left[\int d\phi_n\right], \quad \phi_n = \phi(x_n) \tag{1.28}$$

$$Z_0[J] = \int \mathcal{D}e^{-\frac{i}{2}(\phi,G_0^{-1}\phi)+i(\phi,J)} = \det[G_0]^{1/2}e^{\frac{i}{2}(J,G_0J)} \tag{1.29}$$

$$(J,G_0J) = \int d^4x \int d^4x' J(x)G_0(x,x')J(x') \tag{1.30}$$

$$G_0(x,x') = (\partial_x^2+m^2)^{-1}\delta^{(4)}(x-x') \tag{1.31}$$

## 1.4 Correlation functions and Wick's theorem

By definition of the N-points correlation function

$$\begin{aligned} C(x_1,...,x_N) &= \langle 0|T[\phi(x_1)...\phi(x_N)]|0\rangle \\ &= \frac{\int \mathcal{D}[\phi]\phi(x_1)...\phi(x_N)e^{iS_0[\phi]}}{\int \mathcal{D}[\phi]e^{-S_0[\phi]}} \qquad (1.32) \\ &= \frac{1}{Z_0[J]}(-i)^N \frac{\partial^N Z_0[J]}{\partial J(x_1)...\partial J(x_N)}\bigg|_{J=0} \qquad (1.33) \end{aligned}$$

Feynman propagator

$$D_F(x,y) = \langle 0|T[\phi(x)\phi(y)]|0\rangle = iG_0(x,y) \tag{1.34}$$

Wick's theorem can be derived easily as

$$C(x_1,...,x_N) = \begin{cases} 0, & N-odd \\ \sum_P \Pi'_{i,j} D_F(x_{Pi},x_{Pj}), & N-even \end{cases} \tag{1.35}$$

For example

$$\begin{aligned} C(1,2,3,4) = D_F(1,2)D_F(3,4) + D_F(1,3)D_F(2,4) \\ +D_F(1,4)D_F(2,3) \end{aligned} \tag{1.36}$$

## 1.5 Field interaction and perturbative expansion

Insertion of interaction Lagrangian

$$\mathcal{L}_I = -\mathcal{V}(\phi) \tag{1.37}$$

The generating functional (1.24) becomes

$$Z[J] = \int \mathcal{D}[\phi] e^{iS_0[\phi] - i\int d^4x \mathcal{V}(\phi) + i\int d^4x \phi(x) J(x)} \tag{1.38}$$

$$= e^{-i\int d^4x \mathcal{V}(-i\partial/\partial J)} \int \mathcal{D}[\phi] e^{iS_0[\phi] + i\int d^4x \phi(x) J(x)}$$

$$= e^{-i\int d^4x \mathcal{V}(-i\partial/\partial J)} Z_0[J] \tag{1.39}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 ... \int d^4x_n \mathcal{V}(-i\partial_{J(x_1)}) ... \mathcal{V}(-i\partial_{J(x_n)}) Z_0[J] \tag{1.40}$$

We do have

$$Z^{(0)}[J] = Z_0[J] \tag{1.41}$$

$$Z^{(1)}[J] = -i \int d^4x' \mathcal{V}(-i\partial_{J(x')}) Z_0[J] \tag{1.42}$$

In case of $\phi^4$-interaction

$$\mathcal{V}(\phi) = \lambda\phi^4(x) \to Z^{(1)}[J] = -i\lambda \int d^4x' \left(-i\frac{\partial}{\partial J(x')}\right)^4 Z_0[J] \tag{1.43}$$

$$\to D_F^{(1)}(x,y) = -i\lambda D_F(x,y) \int d^4x' D_F(x',x') D_F(x',x')$$

$$-i\lambda \int d^4dx' D_F(x,x') D_F(x',x') D_F(x',y) \tag{1.44}$$

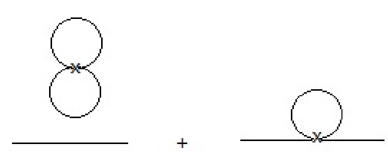

Figure 1.1: First order field propagator in $\phi^4$-interaction.

This shows that $Z[J]$ generate both connected and disconnected Feynman propagators. Let us define the functional

$$iW[J] = \ln Z[J] \to Z[J] = e^{iW[J]} \tag{1.45}$$

$$i\frac{\partial W[J]}{\partial J(x)} = \frac{1}{Z[J]}\frac{\partial Z[J]}{\partial J(x)} \tag{1.46}$$

$$i\frac{\partial^2 W[J]}{\partial J(x)\partial J(y)} = \frac{1}{Z[J]}\frac{\partial^2 Z[J]}{\partial J(x)\partial J(y)} - \frac{1}{Z^2[J]}\frac{\partial Z[J]}{\partial J(x)}\frac{\partial Z[J]}{\partial J(y)} \tag{1.47}$$

So that

$$\begin{aligned} -i\left.\frac{\partial^2 W[J]}{\partial J(x)\partial J(y)}\right|_{J=0} &= \langle\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle & (1.48)\\ &= \langle\phi(x)\phi(y)\rangle_C & (1.49)\end{aligned}$$

It generates the *connected correlation function*, i.e., by definition

$$\begin{aligned}\langle\phi(x_1)...\phi(x_N)\rangle_C &= \langle\phi(x_1)...\phi(x_N)\rangle \\ &\quad - \sum_{Part.(1..k)} (\Pi_i\langle\phi(x_{i1})...\phi(x_{ik})\rangle,\ k < m) \quad (1.50)\end{aligned}$$

From this fact one can write

$$W[J] = i\sum_{n=1}^{\infty}\frac{1}{n!}J(x_1)...J(x_n)\langle\phi(x_1)...\phi(x_n)\rangle_C \tag{1.51}$$

This functional can be used to determine semi-classical approximation in quantum field theory calculation developed by Schwinger. We will look at this later.

## 1.6 Functional integral of fermionic field

### 1.6.1 Grassmann variables

Let $\{\theta_i\}$ be a set of n-Grassmann variables, satisfy the Grassmann algebra

$$\{\theta_i, \theta_j\} = \delta_{ij} \to \theta_i^2 = 0 \tag{1.52}$$

Let $f(\theta)$ be a function of single Grassmann variable theta, its form form will be

$$f(\theta) = a + b\theta, \quad a, b \in R \tag{1.53}$$

From this expression, we can determine the Grassmann calculus, from regular differentiation, we have

$$\frac{d}{d\theta} f(\theta) = \frac{d}{d\theta}(a + b\theta) = b \tag{1.54}$$

$$\text{From Berezin's definition } \int d\theta = 0, \; \int d\theta\theta = 1 \tag{1.55}$$

$$\rightarrow \int d\theta f(\theta) = \int d\theta(a + b\theta) = b \tag{1.56}$$

The Grassmann integral gives a similar result to the differentiation, which look curios. Let us determine the Gaussian function of 2n-Grassmann variables

$$\begin{aligned} G[A] = \exp\left\{\sum_{i,j} \bar{\theta}_i A_{ij} \theta_i\right\} &= \left(1 + \sum_{i,j} \bar{\theta}_i A_{ij} \theta_j\right) \\ &= \Pi_i \left(1 + \bar{\theta}_i \sum_j A_{ij} \theta_j\right) \end{aligned} \tag{1.57}$$

$$\rightarrow I[A] = \Pi_i \left[\int d\theta_i d\bar{\theta}_i\right] G[A] = \det[A] \tag{1.58}$$

Generalize with the linear terms

$$G[A, b, \bar{b}] = \exp\left\{\sum_{i,j} \bar{\theta}_i A_{ij} \theta_j + \sum_i (\bar{b}_i \theta_i + \bar{\theta}_i b_i)\right\} \tag{1.59}$$

$$\rightarrow \theta_i \rightarrow \theta'_i = \theta_i + \sum_j (A^{-1})_{ij} b_j \text{ also for } \bar{\theta}_i \tag{1.60}$$

$$\rightarrow G[A, b] = \exp\left\{\sum_{i,j} \bar{\theta}'_i A_{ij} \theta'_j + \sum_{ij} \bar{b}_i (A^{-1})_{ij} b_j\right\} \tag{1.61}$$

$$\rightarrow I[A, b] = \Pi_i \left[\int d\theta'_i d\bar{\theta}'_i\right] G[A, b] = \det[A] e^{+\sum_{ij} \bar{b}_i (A^{-1})_{ij} b_j} \tag{1.62}$$

We can simply proof this by written all expressions in matrix form.

### 1.6.2 Fermionic generating functional

Free fermionic field action functional, i.e., Dirac action functional, is

$$S_0[\psi, \bar{\psi}] = \int d^4x \bar{\psi}(x)(i\partial\!\!\!/_x - m)\psi(x) \tag{1.63}$$

$$\equiv \int d^4x d^4y \bar{\psi}(x) G_0(x,y)^{-1}\psi(y) \equiv (\bar{\psi}, G_0^{-1}\psi) \tag{1.64}$$

$$G_0(x,y) = (i\partial\!\!\!/_x - m)^{-1}\delta^{(4)}(x-y) \tag{1.65}$$

The generating functional is defined in the form

$$Z_0[J, \bar{J}] = \int \mathcal{D}[\psi, \bar{\psi}] e^{iS_0[\psi,\bar{\psi}] + i(\bar{\psi}, J) + i(\bar{J}, \psi)} \tag{1.66}$$

With the fermionic quantum field, i.e. $\{\psi, \bar{\psi}\} = 1$, then $Z_0[J, \bar{J}]$ is Grassmannian Gaussian integral. Such that

$$Z_0[J, \bar{J}] = \det[iG_0^{-1}] e^{i(\bar{J}, G_0 J)} \tag{1.67}$$

For example

$$G_0(y,x) = \frac{-i}{Z_0[0,0]} \frac{\partial^2 Z_0[J, \bar{J}]}{\partial J(x) \partial \bar{J}(y)}\bigg|_{J=\bar{J}=0} \tag{1.68}$$

For the case of interaction, the analysis can be done similar to the bosonic generating function. In case of Yukawa-type interaction, the combined bosonic and generating functional can be defined and used for generating the perturbative expansion of the interacting S-matrix.