2 Gauge Fields and Functional Quantization

2.1 Gauge symmetry

2.1.1 Gauge transformation

For a vector field $A^\mu(x)$, its field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and field Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.1}$$

is invariant under a *gauge transformation* (GT)

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \alpha \tag{2.2}$$

where $\alpha(x)$ is any real scalar function. The gauge invariant matter-coupling gauge field theory, i.e. with the complex scalar field $\phi(x)$, can be modeled with the Lagrangian

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.3}$$

$$\text{Covariant derivative}: \quad D_\mu \phi = (\partial_\mu + igA_\mu)\phi \tag{2.4}$$

$$\to F_{\mu\nu} = \frac{-i}{g}[D_\mu, D_\nu] \tag{2.5}$$

$$\text{GT}: \quad A_\mu(x) \to A_\mu(x) + \frac{1}{g}\partial_\mu \alpha(x), \;\; \phi(x) \to e^{i\alpha(x)}\phi(x) \tag{2.6}$$

$$\phi^* \phi \to \phi^* \phi, \text{ and } F_\mu \to F_{\mu\nu} \tag{2.7}$$

$$D_\mu \phi \to (\partial_\mu + igA_\mu + i\partial_\mu \alpha) e^{i\alpha}\phi = e^{i\alpha}(D_\mu \phi)$$

$$\text{So that } (D_\mu \phi)^* (D^\mu \phi) \to (D_\mu \phi)^* (D^\mu \phi) \tag{2.8}$$

This is how *gauge symmetry* play its role in the matter-coupled gauge field theory.

2.1.2 Gauge groups and their algebras

Let G be a gauge group, with its generator $g = e^{i\alpha}$, with $g^\dagger g = 1$. We can rewrite the gauge transformation in term of the action of g as

$$\phi \to g\phi, \text{ and } A_\mu \to g^\dagger (A_\mu + \frac{i}{g}\partial_\mu) g \tag{2.9}$$

The gauge group G can be classified to be

- The simplest *local* $U(1)$ group when $\alpha(x)$ is any real scalar function. It will be called global $U(1)$ for the case of α is being a real constant. The gauge transformation on field can be done for any complex-valued field function.

- The local $SU(N)$ group when $\alpha(x)$ is a matrix-valued function, i.e., the square matrix. Its expansion of the basis is written in the form

$$\alpha(x) = \alpha_a(x)t^a, \ a = 1, 2, ..., dim(N) \tag{2.10}$$

where $\dim(N) = N^2 - 1$ is dimension of $N \times N$ square matrix space, i.e., where $\{t^a\}$ is its basis set. Noe that $\{t^a\}$ is defined to be the hermitian square matrices, with the basic properties of $Tr(t^a) = 0$ and $Tr(t^a t^b) = 2\delta^{ab}$

The algebra of $N > 1$ gauge group is determined from the commutation relation

$$[t^a, t^b] = if^{abc}t^c, \ \{t^a, t^b\} = \frac{1}{N}\delta^{ab} + d^{abc}t^c \tag{2.11}$$

where f^{abc} is known as its *structure constants.* The *irreducible* representation of G , in terms of $N \times N$ matrix, is constructed from the eigen-basis of Casimir oeprator(s) that constructed from a set o basis $\{t^a\}$. The *adjoint* representation of G, the set of generators constructed form the structure constants in form of $(N^2 - 1) \times (N^2 - 1)$ matrices, as

$$(t^a)^{bc} = if^{abc} \tag{2.12}$$

The gauge field is classified according to it gauge group to be

- abelian gauge field for a gauge group of $U(1)$
- non-abelian gauge field for a gauge group of $SU(N)$, with $N > 1$

2.1.3 Gauge fixing condition

In order to get physical prediction of the gauge field $A_\mu(x)$, i.e., the measurable value of the corresponding electric and magnetic fields, we have to kill its gauge symmetry, by assigning the *gauge fixing* condition. A general form of gauge fixing condition is denoted in functional form as

$$G[A_\mu] = 0 \tag{2.13}$$

2.2 Functional quantization of abelian gauge field

For a local U(1) gauge field $A^\mu(x)$, its action functional is

$$\begin{aligned} S_0[A_\mu] = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{2}\int d^4x \partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= +\frac{1}{2}\int d^4x A^\nu(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)A^\mu \\ &= \frac{1}{2}\int d^4x d^4y A^\mu(x)\Delta^{-1}_{\mu\nu}(x,y)A^\nu(y) \end{aligned} \tag{2.14}$$

$$\text{where} \ \ \Delta^{-1}_{\mu\nu}(x,y) = (\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)_x \delta^{(4)}(x-y) \tag{2.15}$$

The generating functional is written in the form

$$Z_0[J] = \int \mathcal{D}[A_\mu] e^{iS_0[A_\mu]+i\int d^4x J_\mu(x)A^\mu(x)} \tag{2.16}$$

The gauge fixing condition is need to be inserted into this expression in order to kill the gauge symmetry of a gauge field $A_\mu(x)$.

Since gauge transformation is a kind of unitary transformation U, so that let us denote the gauge field with gauge symmetry as $A_\mu^U(x)$, it is said to be a configuration of gauge field on *gauge manifold*. Following Faddeev and Popov, let us define the integral

$$\Delta_G^{-1}[A_\mu] = \int \mathcal{D}[U]\delta(G[A_\mu]) \tag{2.17}$$

This expression means that the construction of $\Delta_G^{-1}[A_\mu]$ is done under the constraint condition $G[A_\mu] = 0$.

Let us show that $\Delta_G^{-1}[A_\mu]$ itself is gauge invariant. Let us assign a gauge transformation $U \to UU'$, so that

$$\mathcal{D}[U] \to \mathcal{D}[UU'] = \mathcal{D}[U] \quad \textit{Haar measure} \tag{2.18}$$

$$\begin{aligned} \to \Delta_D^{-1}[A_\mu^{U'}] &= \int \mathcal{D}[U]\delta(G[A_\mu^{UU'}]), \quad U'' = UU' \\ &= \int \mathcal{D}[U'']\delta(G[A_\mu^{U''}]) = \Delta_G^{-1}[A_\mu] \end{aligned} \tag{2.19}$$

From (2.17), one can write

$$1 = \Delta_G[A_\mu] \int \mathcal{D}[U]\delta(G[A_\mu^U]) \tag{2.20}$$

Insertion into (2.16) can be consistently as

$$\begin{aligned} Z_0[J] &= \int \mathcal{D}[A_\mu] \times 1 \times e^{iS[A_\mu, J_\mu]} \\ &= \int \mathcal{D}[A_\mu]\Delta_G[A_\mu] \int \mathcal{D}[U]\delta(G[A_\mu^U])e^{iS[A_\mu, J_\mu]} \end{aligned} \tag{2.21}$$

Noe make a change of variable $A_\mu \to A_\mu^{U'}$, we will have from (2.21)

$$Z_0[J] = \int \mathcal{D}[U] \int \mathcal{D}[A_\mu^{U'}] e^{iS[A_\mu^{U'}, J_\mu]} \Delta_G[A_\mu^{U'}]\delta(G[A_\mu^{U'U}]) \tag{2.22}$$

Now assign $U' = U^1$, and make use of the gauge invariant of $\mathcal{D}[A_\mu], S[A_\mu, J_\mu]$ and $\Delta_G[A_\mu]$, we will have

$$Z_0[J] = \left[\int \mathcal{D}[U]\right] \int \mathcal{D}[A_\mu]\Delta_G[A_\mu]\delta(G[A_\mu])e^{iS[A_\mu, J_\mu]} \tag{2.23}$$

Note that the factor $\int \mathcal{D}[U]$ is actually infinite, but trivially deal to nothing in quantum field calculation.

We now compute the factor $\Delta_G[A_\mu]$, by using the condition $G[A_\mu]$, as

$$\mathcal{D}[U] = \mathcal{D}[G]\left|\frac{\delta U}{\delta G}\right| \tag{2.24}$$

$$\to \Delta_G^{-1}[A_\mu] = \int \mathcal{D}[U]\delta(G[A_\mu^U]) = \int \mathcal{D}[G]\left|\frac{\delta U}{\delta G}\right|\delta(G) = \left|\frac{\delta U}{\delta G}\right|_{G=0} \tag{2.25}$$

$$\to \Delta_G[A_\mu] = \left|\frac{\delta G}{\delta U}\right|_{G=0} \tag{2.26}$$

For example

$$U = e^{i\phi(x)},\ G[A_\mu^U] = \partial_\mu(A^\mu + \partial^\mu\phi) = \partial_\mu A^\mu + \partial^2\phi \tag{2.27}$$

$$\frac{\delta G(x)}{\delta\phi(y)} = \partial^2\delta(x-y) \to \Delta_G[A_\mu] = \det|\partial^2| \tag{2.28}$$

which is a constant independent of A_μ. Next let us modify the gauge fixing condition to be $G[A_\mu] = c(x)$, where $c(x)$ is arbitrary scalar function of x. Now from (2.23), we will have

$$Z_0[J] \quad \sim \quad \int \mathcal{D}[A_\mu\delta(G[A_\mu] - c(x))e^{iS[A,J]} \tag{2.29}$$

We now average over arbitrary function $c(x)$ with a Gaussian weight as

$$\begin{aligned} Z_0[J] &= \mathcal{N}\int \mathcal{D}[c]e^{-i\int d^4x c^2(x)/2\alpha}\int \mathcal{D}[A_\mu]\delta(G[A_\mu] - c(x))e^{iS[A,J]} \\ &= \mathcal{N}\int \mathcal{D}[A_\mu]e^{iS[A,J]}e^{-i\int d^4x G^2[A_\mu]/2\alpha} \end{aligned} \tag{2.30}$$

For example of $G[A_\mu] = \partial_\mu A^\mu$, the gauge field action functional will appear in the form

$$\begin{aligned} S_0[A_\mu] &= \frac{1}{2}\int d^4x A^\mu(x)(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu(x) - \frac{1}{2\alpha}\int d^4x(\partial_\mu A^\mu)^2(x) \\ &= \frac{1}{2}\int d^4x A^\mu(x)\left(\partial^2 g_{\mu\nu} - \frac{\alpha-1}{\alpha}\partial_\mu\partial_\nu\right)A^\nu(x) \qquad (2.31) \\ &= \frac{1}{2}\int d^4x d^4y A^\mu(x)\Delta^{-1}(x,y;\alpha)_{\mu\nu}A^\nu(y) \qquad (2.32) \end{aligned}$$

$$\Delta^{-1}(x,y;\alpha)_{\mu\nu} = \left(\partial^2 g_{\mu\nu} - \frac{\alpha-1}{\alpha}\partial_\mu\partial_\nu\right)\delta^{(4)}(x-y) \tag{2.33}$$

And finally we get

$$Z_0[J] = \mathcal{N}' e^{\frac{i}{2}\int d^4x d^4y J^\mu(a)\Delta_{\mu\nu}(x,y;\alpha)J^\nu(y)} \tag{2.34}$$

with a gauge fixing parameter α.

2.3 Functional quantization of non-abelian gauge field

2.4 BRST symmetry