2 Gauge Fields and Functional Quantization

2.1 Lie groups and Lie algebras

2.2 Abelian gauge field

2.2.1 Generating functional

At the end, after applying gauge fixing condition into the functional integral formulation of the generating functional $Z_0[J]$, we have derived an its exact solution in the form

$$Z_0[J] = Z_0[0] e^{i \int d^4x \int d^4y J^\mu(x) G^0_{\mu\nu}(x,y) J^\nu(y)} \tag{2.1}$$

$$G^0_{\mu\nu}(x,y) = \left(g^{\mu\nu} \partial_x^2 - \frac{\alpha - 1}{\alpha} \partial^\mu \partial^\nu \right)^{-1} \delta^{(4)}(x-y) \tag{2.2}$$

In quantum field theory, this Green's function is derived from an expression

$$G^0(x,y)_{\mu\nu}(x,y) = i \langle x | T[A_\mu(x) A_\nu(y)] | y \rangle \tag{2.3}$$

2.2.2 Field propagator

Classically, the Green's function (2.3) satisfy the equation

$$\left(g^{\mu\nu} \partial_x^2 - \frac{\alpha - 1}{\alpha} \partial^\mu \partial^\nu \right) G^0_{\nu\lambda}(x,y) = \delta^\mu{}_\lambda \delta^{(4)}(x-y) \tag{2.4}$$

Its solution is derived within a few following steps

$$G^0_{\mu\nu}(x,y) = \int \frac{d^4p}{(2\pi)^4} G^0_{\mu\nu}(p) e^{-ip\cdot(x-y)} \tag{2.5}$$

$$(2.4) \to \left(g^{\mu\nu} p^2 - \frac{\alpha - 1}{\alpha} p^\mu p^\nu \right) G^0_{\mu\nu}(p) = -1 \tag{2.6}$$

$$G^0_{\mu\nu}(p) = -\frac{1}{p^2} \left(g_{\mu\nu} + (\alpha - 1) p_\mu p_\nu \right) \tag{2.7}$$

The field propagator is derived in the form

$$\Delta^0_{\mu\nu}(p) = i G^0_{\mu\nu}(p) = \frac{-i}{p^2} \left(g_{\mu\nu} + (\alpha - 1) \frac{p_\mu p_\nu}{p^2} \right) \tag{2.8}$$

where α is known as Feynman-t'Hooft gauge fixing parameter:

- Feynman gauge, $\alpha = 1$,

$$G^0_{\mu\nu}(p) = -\frac{-i g_{\mu\nu}}{p^2}$$

- Landau gauge, $\alpha = 0$,

$$G^0_{\mu\nu}(p) = \frac{-i}{p^2} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)$$

  Note that the bracket on the RHS is called *transverse projection operator*.

2.3 Non-abelian gauge field

Let $A_\mu(x)$ be non-abelian gauge field

$$D_\mu = \partial_\mu + igA_\mu(x) \tag{2.9}$$

$$\to F_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\mu A_\nu + ig[A_\mu, A_\nu] \tag{2.10}$$

Let us denote $A_\mu(x) = A^a_\mu(x)t^a$, and $[t^a, t^b] = if^{abc}t^c$, we will have $F_{\mu\nu} = F^a_{\mu\nu}t^a$ where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc}A^b_\mu A^c_\nu \tag{2.11}$$

Note that $\{t^a\}$ is a set of generators of SU(N) gauge group and f^{abc} is called structure constant.

Under a gauge transformation $U = e^{ig\alpha^a(x)t^a}$, the gauge field will transformed as

$$A_\mu(x) \xrightarrow{U} A^U_\mu = U^{-1}A_\mu U + \frac{i}{g}U^{-1}\partial_\mu U = A_\mu + \delta A_\mu \tag{2.12}$$

$$U \simeq 1 + \frac{i}{g}\alpha^a(x)t^a + ... \tag{2.13}$$

$$\to \delta A^a_\mu t^a = i\alpha^b(x)[t^b, t^c]A^c_\mu - \partial_\mu \alpha^a t^a + ... \tag{2.14}$$

$$\delta A^a_\mu(x) = -f^{abc}\alpha^b(x)A^c_\mu(x) - \partial_\mu \alpha^a(x) + ... \tag{2.15}$$

$$\frac{\delta A^a_\mu(x)}{\delta \alpha^b(y)} = -\left[\partial_\mu \delta^{ab} + f^{abc}A^c_\mu\right]\delta(x-y) \equiv -D^{ab}_\mu[A]\delta(x-y) \tag{2.16}$$

$$\to D^{ab}_\mu[A] = \delta^{ab}\partial_\mu + f^{abc}A^c_\mu \tag{2.17}$$

We have derived the *adjoin representation* of covariant derivative.

The classical action of non-abelian gauge field is

$$S[A] = -\frac{1}{2g^2}\int d^4x Tr[F_{\mu\nu}F^{\mu\nu}] = -\frac{1}{4g^2}\int d^4x F^a_{\mu\nu}F^{a\mu\nu} \tag{2.18}$$

The Faddeev-Popov generating functional of the gauge field is written as

$$Z_0[J] = \int \mathcal{D}[A]e^{iS[A,J]}\delta(G[A])\Delta_F P[A] \tag{2.19}$$

$$G[A] = G^a[A]t^a,\ G^a[A] = \partial^\mu A^a_\mu + c^a(x) = 0 \tag{2.20}$$

It is the gauge fixing condition. $\Delta_{FP}[A]$ is Faddeev-Popov determinant, it is evaluated as

$$\Delta_{FP}[A] = \det\left(\frac{\delta G}{\delta \alpha}\right) = \det\left(\frac{\partial G^a}{\partial A^c_\mu}\frac{\partial A^c_\mu}{\partial \alpha_b}\right) \tag{2.21}$$

$$\begin{aligned} M_{FP} = \frac{\partial G}{\partial A^c_\mu}\frac{\partial A^c_\mu}{\partial \alpha} \to \langle x, a|M_{FP}|y, b\rangle &= \int_z \frac{\partial G^a(x)}{\partial A^c_\mu(z)}\frac{\partial A^c_\mu(z)}{\partial \alpha^b(y)} \\ &= -\int_z \frac{\partial G^a(x)}{\partial A^c_\mu(z)}D^{cb}_\mu(z-y) \end{aligned} \tag{2.22}$$

From (2.20), we have

$$\frac{\partial G^a(x)}{\partial A^c_\mu(z)} = \delta^{ac}\partial^\mu\delta(x-z) \tag{2.23}$$

$$(2.22) \to \langle x, a|M_{FP}|y, b\rangle = -\int_z \delta^{ac}\delta(x-z)\partial^\mu_z D^{cb}_\mu(z-y)$$

$$= -\partial^\mu D^{ab}_\mu(x-y) \tag{2.24}$$

$$(2.21) \to \Delta_{FP} = -\det(\partial^\mu D_\mu[A]) \tag{2.25}$$

We can evaluate the det(...) by using Gaussian integral of fermionic (scalar Grassmann-valued) *ghost* fields $\eta_a(x)$ and $\bar{\eta}_a(x)$ as

$$\det(\partial^\mu D_\mu[A]) = \int \mathcal{D}[\eta,\bar{\eta}]e^{i\int d^4x\bar{\eta}_a(x)\partial^\mu D^{ab}_\mu[A]\eta_b(x)} \tag{2.26}$$

Similar to the case of abelian gauge field, the gauge fixing parameter λ is inserted by doing the Gaussian average of the functional integral over all gauge condition, finally we will have an expression of the functional integral in the form

$$Z_0[0] = \int \mathcal{D}[A]\mathcal{D}[\eta,\bar{\eta}]e^{i\int d^4x\mathcal{L}_{YM}[A,\eta,\bar{\eta}]} \tag{2.27}$$

$$\mathcal{L}_{YM}[A,\eta,\bar{\eta}] = -\frac{1}{2g^2}tr(F_{\mu\nu}F^{\mu\nu}) + \frac{\lambda}{2g^2}(\partial_\mu A^\mu)^2 - \bar{\eta}\partial_\mu D^\mu[A]\eta \tag{2.28}$$

$$= -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2g^2}(\partial_\mu A^{a\mu}(\partial_\nu A^{a\nu}) - \bar{\eta}^a\partial_\mu D^{ab}[A]\eta^b \tag{2.29}$$

This pure gauge field Lagrangian is not quadratic form, anyway we can treat it in perturbative and non-perturbative aspects within the quantum theory of this field.

2.4 BRST symmetry

Let us consider the matter-coupled non-abelian gauge field Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^a_{\mu\nu}A^{a\mu\nu} - \frac{1}{2}B_aB_a + B_a\partial^\mu A^a_\mu - \bar{\eta}^a\partial^\mu D^{ab}_\mu[A]\eta^b \tag{2.30}$$

where ψ is Dirac spinor field and B is auxiliary *Hubbard-Stratonovich* field, which has no dynamics by its own.

Becchi, Rouet, and Stora (1974, 1976), and Tyutin (2008) realized that this Lagrangian has *BRST* symmetry under the following transformations

$$\delta A^a_\mu = \epsilon D^{ab}_\mu\eta_b \tag{2.31}$$

$$\delta\psi = ig\epsilon\eta^a t^a\psi \tag{2.32}$$

$$\delta\eta^a = -\frac{1}{2}g\epsilon f^{abc}\eta_b\eta_c \tag{2.33}$$

$$\delta\bar{\eta}^a = \epsilon B^a \tag{2.34}$$

$$\delta B^a = 0 \tag{2.35}$$

Note that equations (2.31,2.32) are local gauge transformations, together with (2.35) this shows invariant of the first three terms in $\mathcal{L}$. Equations (2.33,2.34) are assigned to get a cancellation of the changes of A_μ in the fourth and fifth terms of $\mathcal{L}$.

The BRST si the global symmetry of the gauge-fixed action. The existence of this symmetry leads to the renormalizability of non-abelian gauge field, i.e, the UV divergence can be eliminated by the process of *renormalization*. Looking for more discussion from argument given Dr.Vadim Kaplunovsky at URL (https://web2.ph.utexas.edu/ vadim/Classes/2008f/brst.pdf).

2.5 Gauge field phase factor (Wilson loop)