AQFT: non-perturbative method
Thursday 4, February 2021, lecture 3

3 Quantum Chromodynamics or QCD

3.1 QCD Lagrangian

QCD is a gauge theory of quark (q)-gluon(g) interaction, with SU(3) color gauge group. The QCD Lagrangian, with gauge fixing and FP ghost terms, is

$$\mathcal{L}_{QCD} = \bar{\psi}^i(i\not{D}^{ij} - m\delta^{ij})\psi^i - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\lambda}(\partial A^a)^2 - \bar{\eta}^a\partial^\mu D^{ab}_\mu[A]\eta^b \quad (3.1)$$

where ψ^i is quark color triplet, and [1]

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + q_c f^{abc} A^b_\mu A^c_\nu \quad (3.2)$$

$$D^{ij}_\mu = \delta^{ij}\partial_\mu - iq_c(t^a)^{ij}A^a_\mu \quad (3.3)$$

$$\text{Adj. rep.} \rightarrow D^{ab}_\mu = \delta^{ab}\partial_\mu - q_c f^{abc} A^c_\mu \quad (3.4)$$

Note that q_c is the color charge, or gauge coupling constant, and η is FP ghost field.

3.2 QCD Feynman rules

Let us determine

$$\begin{aligned}
-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + qf^{abc}A^b_\mu A^c_\nu) \\
&\quad \times(\partial^\mu A^{a\nu} - \partial^\nu A^{a\nu} + qf^{ade}A^{d\mu}A^{e\nu}) \\
&= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\nu}) \\
&\quad -\frac{1}{2}q_c f^{abc}A^b_\mu A^c_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\nu}) \\
&\quad -\frac{1}{4}q_c^2 f^{abc}F^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}
\end{aligned} \quad (3.5)$$

We can rewrite (3.1) in the form

$$\mathcal{L}_{QCD} = \mathcal{L}_q + \mathcal{L}_{g+gf} + \mathcal{L}_{q-g} + \mathcal{L}_{2g} + \mathcal{L}_{3g} + \mathcal{L}_{ghost} + \mathcal{L}_{g-ghost} \quad (3.6)$$

[1]Adjoin representation $(t^a)^{bc} = -if^{abc}$.

where

$$
\begin{aligned}
\mathcal{L}_q &= \bar{\psi}^i (i\partial\!\!\!/ - m)^{ij} \psi^j && (3.7)\\
\mathcal{L}_{g+gf} &= \frac{1}{2} A_\mu \left[g^{\mu\nu} \partial^2 - \left(1 - \frac{1}{\lambda}\right) \partial^\mu \partial^\nu \right] A_\nu && (3.8)\\
\mathcal{L}_{q-g} &= q_c \bar{\psi}^i \gamma^\mu \psi^j (t^a)^{ij} A^a_\mu && (3.9)\\
\mathcal{L}_{2g} &= -\frac{1}{2} q_c f^{abc} A^b_\mu A^c_\nu (\partial^\mu A^{a\nu} - \partial^\nu A^{a\nu}) && (3.10)\\
\mathcal{L}_{3g} &= \frac{1}{4} q_c^2 f^{abc} F^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} && (3.11)\\
\mathcal{L}_{ghost} &= \bar{\eta}^a \delta^{ab} \partial^2 \eta^b && (3.12)\\
\mathcal{L}_{g-ghost} &= -q_c f^{abc} (\partial^\mu \bar{\eta}^a) \eta^b A^c_\mu && (3.13)
\end{aligned}
$$

Feynman rules are applied to these Lagrangian as in the figures (3.1, 3.2).

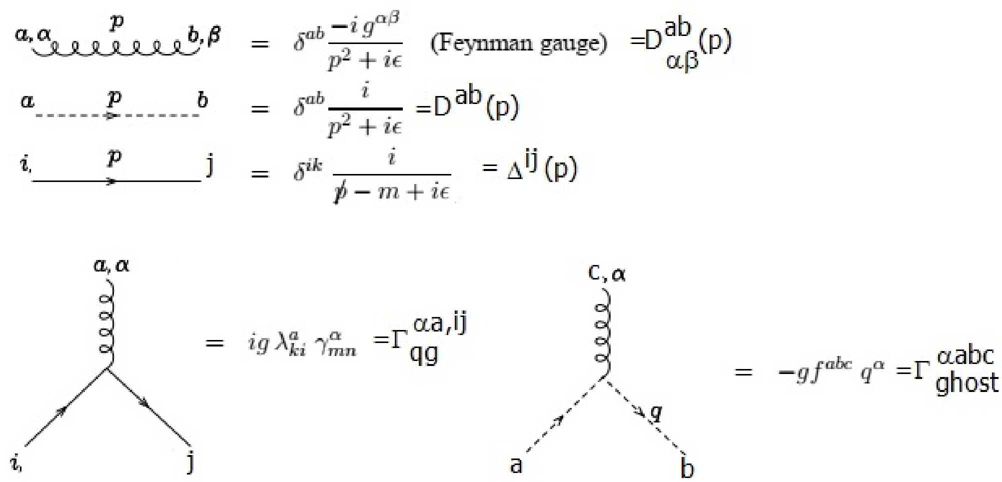
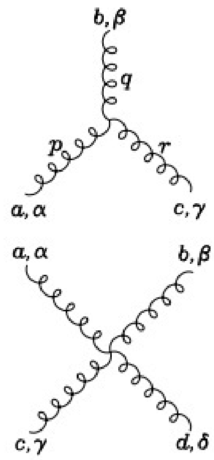


Figure 3.1: QCD Feynman rules.



$$= gf^{abc}\left[g^{\alpha\beta}(p-q)^{\gamma} + g^{\beta\gamma}(q-r)^{\alpha} + g^{\gamma\alpha}(r-p)^{\beta}\right] = \Gamma^{abc}_{\alpha\beta\gamma}$$

$$\begin{aligned} = & -ig^2 f^{xac} f^{xbd}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) = \Gamma^{abcd}_{\alpha\beta\gamma\delta} \\ & -ig^2 f^{xad} f^{xbc}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) \\ & -ig^2 f^{xab} f^{xcd}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \end{aligned}$$

Figure 3.2: QCD gluon vertices.

3.3 Elements of su(3) algebra

The su(3) generators are $t^1 = \frac{1}{2}\lambda^a,\ a = 1, 2, ..., 8,$ while $\{\lambda^a\}$ is a set of Gell-Mann matrices. They satisfy the Lie algebra

$$\left[t^a, t^b\right] = if^{abc}t^c \tag{3.14}$$

where f^{abc} are structure constants of the algebra, i.e. they are totally antisymmetric. The generator t^a and structure constant f^{abc} satisfy Jacobi identity

$$\left[t^a, \left[t^b, t^c\right]\right] + \left[t^c, \left[t^a, t^b\right]\right] + \left[t^b, \left[t^c, t^a\right]\right] = 0 \tag{3.15}$$

$$f^{bcd}f^{ade} + f^{abd}f^{cde} + f^{cad}f^{bde} = 0 \tag{3.16}$$

And there are related by

$$f^{abc} = -2iTr\left[t^a, t^b\right]t^c \tag{3.17}$$

Summary of some color algebra always used in quark-gluon scattering amplitude calculation.

$$\left[t^a, t^b\right] = if^{abc}t^c \tag{3.18}$$

$$Tr[t^a] = 0,\ Tr[t^a t^b] = \frac{1}{2}\delta^{ab} \tag{3.19}$$

$$\sum_{a,j}(t^a)_{ij}(t^b)_{jk} = N_F\delta_{ik},\ \sum_{a,c}(f^a)_{bc}(f^a)_{cd} = N_A\delta_{bd} \tag{3.20}$$

The representations of the generator t^a consist of i) fundamental representation $(t^a)_{ij}$ in the color triplet ψ^i, and ii) adjoin representation $(t^a)^{bc} = -if^{abc}$ on the color group basis. The basic properties of the generator t^a are

$$Tr[t^a] = 0, \quad Tr[t^a t^b] = \frac{1}{2}\delta^{ab} \tag{3.21}$$

$$\sum_{a,j} (t^a)_{ij}(t^a)_{jk} = C_F \delta_{ik}, \quad C_F = \frac{N^2-1}{2N} = \frac{4}{3}, \; N = 3 \tag{3.22}$$

$$\sum_{b,c} f^{abc} f^{dbc} = C_A \delta^{ad}, \; C_A = N = 3, \; N = 3 \tag{3.23}$$

3.4 QCD elementary processes

3.4.1 $qq \to qq$ scattering

Feynman diagrams

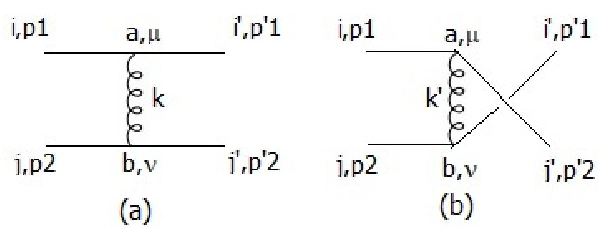


Figure 3.3: $qq \to qq$ scattering diagrams.

The amplitudes are

$$\begin{aligned} iM_a &= (-iq_c)^2 \bar{U}_{i'}(1')\gamma^\mu t^a_{i'i} U_i(1) \frac{-ig_{\mu\nu}}{k^2} \bar{U}_{j'}(2')\gamma^\nu t^b_{j'j} U_j(2) \\ \to M_a &= i\frac{q_c^2}{k^2}\left(t^a_{i'i} t^b_{j'j}\right)[\bar{U}_{i'}(1')\gamma^\mu U_i(1)][\bar{U}_{j'}(2')\gamma_\mu U_j(2)] \end{aligned} \tag{3.24}$$

$$\begin{aligned} iM_b &= (-iq_c)^2 \bar{U}_{j'}(2')\gamma^\mu t^a_{j'i} U_i(1) \frac{-ig_{\mu\nu}}{k'^2} \bar{U}_{i'}(1')\gamma^\nu U_j(2) \\ \to M_b &= i\frac{q_c^2}{k'^2}\left(t^a_{j'i} t^b_{i'j}\right)[\bar{U}_{j'}(2')\gamma^\mu U_i(1)][\bar{U}_{i'}(1')\gamma_\mu U_j(2)] \end{aligned} \tag{3.25}$$

Total amplitude, and the amplitude squared are

$$M = M_a + M_b \to |M|^2 = |M_a|^1 + |M_b|^2 + 2M_a^* M_b \tag{3.26}$$

Let us determine

$$\begin{aligned} |M_a|^2 &= \frac{q_c^4}{s^2} \left(t^a_{i'i} t^b_{j'j}\right) \left(t^a_{i'i} t^b_{j'j}\right)^\dagger [\bar{U}_{i'}(1')\gamma^\mu U_i(1)][\bar{U}_{j'}(2')\gamma_\mu U_j(2)] \\ &\quad \times [\bar{U}_i(1)\gamma^\nu U_{i'}(1')][\bar{U}_j(2)\gamma_\nu U_{j'}(2')] \end{aligned} \tag{3.27}$$

Since

$$\left(t^a_{i'i} t^b_{j'j}\right) \left(t^a_{i'i} t^b_{j'j}\right)^\dagger = \left(t^a_{i'i} t^b_{j'j}\right) \left(t^b_{jj'} t^a_{ii'}\right) = \left(t^a_{i'i} t^a_{ii'}\right) \left(t^b_{j'j} t^b_{jj'}\right) = \frac{4}{3} \cdot \frac{4}{3}$$

Then we have

$$\begin{aligned} |M_a|^2 &= \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{q_c^4}{t^2} [\bar{U}_{i'}(1')\gamma^\mu U_i(1) \bar{U}_i(1)\gamma^\nu U_{i'}(1')] \\ &\quad \times [\bar{U}_{j'}(2')\gamma_\mu U_j(2) \bar{U}_j(2)\gamma_\nu U_{j'}(2')] \end{aligned} \tag{3.28}$$

When averaged overall incoming states ans sum overall outgoing states, we will have

$$\begin{aligned} \overline{|M_a|^2} &= \frac{1}{4} \sum_{s_{i'} s_{j'}} |M_a|^2 \\ &= \frac{4}{9} \frac{q_c^4}{t^2} Tr\left[\gamma^\mu (\not{p}_1 - m)\gamma^\nu (\not{p}_{1'} - m)\right] Tr\left[\gamma_\mu (\not{p}_2 - m)\gamma_\nu (\not{p}_{2'} - m)\right] \end{aligned} \tag{3.29}$$

We will also have

$$\begin{aligned} \overline{|M_b|^2} &= \frac{1}{4} \sum_{s_{i'} s_{j'}} |M_b|^3 \\ &= \frac{4}{9} \frac{q_c^4}{u^2} Tr\left[\gamma^\mu (\not{p}_2 - m)\gamma^\nu (\not{p}_{1'} - m)\right] Tr\left[\gamma_\mu (\not{p}_1 - m)\gamma_\nu (\not{p}_{2'} - m)\right] \end{aligned} \tag{3.30}$$

Next let us determine

$$\begin{aligned} M_a^* M_b &= \frac{q_c^4}{tu} \left(t^a_{i'i} t^b_{j'j}\right)^\dagger \left(t^a_{j'i} t^b_{i'j}\right) \left[\bar{U}_i(1)\gamma^\mu U_{i'}(1')\right] \left[\bar{U}_j(2)\gamma_\mu U_{j'}(2')\right] \\ &\quad \times \left[\bar{U}_{j'}(2')\gamma^\nu U_i(1)\right] \left[\bar{U}_{i'}(1')\gamma_\nu U_j(2)\right] \end{aligned} \tag{3.31}$$

Since

$$\left(t^a_{i'i}t^b_{j'j}\right)^\dagger\left(t^a_{j'i}t^b_{i'j}\right) = t^b_{jj'}t^a_{j'i}t^a_{ii'}t^b_{i'j} = \frac{1}{6}\delta^{ab}\delta_{ij}\frac{1}{6}\delta^{ab}\delta_{ij} = \frac{1}{36}\cdot 8\cdot 3 = \frac{2}{3}(?)$$

We then have

$$M^*aM_b = \frac{2}{3}\cdot\frac{q_c^4}{tu}\left[\bar{U}_i(1)\gamma^\mu U_{i'}(1')\bar{U}_{i'}(1')\gamma_\nu U_j(2)\right.$$
$$\left.\ldots\ \bar{U}_j(2)\gamma_\mu U_{j'}(2')\bar{U}_{j'}(2')\gamma^\nu U_i(1)\right] \tag{3.32}$$

When averaged overall incoming states ans sum overall outgoing states, we will have

$$\overline{M_a^* M_b} = \frac{1}{4}\sum_{s_{i'}s_{j'}} M_a^* M_b$$
$$= \frac{1}{6}\cdot\frac{q_c^4}{tu}Tr\left[\gamma^\mu(\not{p}_{1'}-m)\gamma^\nu(\not{p}_2-m)\gamma_\mu(\not{p}_{2'}-m)\gamma_\nu(\not{p}_1-m)\right] \tag{3.33}$$

For the massless (energetic) quark, we will have

$$\overline{|M_a|^2} = \frac{4}{9}\frac{q_c^4}{t^2}Tr[\gamma^\mu\not{P}_1\gamma^\nu\not{p}_{1'}]Tr[\gamma_\mu\not{p}_2\gamma_\nu\not{p}_{2'}] \tag{3.34}$$
$$\overline{|M_b|^2} = \frac{4}{9}\frac{q_c^4}{u^2}Tr[\gamma^\mu\not{P}_2\gamma^\nu\not{p}_{1'}]Tr[\gamma_\mu\not{p}_1\gamma_\nu\not{p}_{2'}] \tag{3.35}$$
$$2\overline{M_a^* M_b} = \frac{1}{3}\frac{q_c^4}{tu}Tr[\gamma^\mu\not{p}_{1'}\gamma^\nu\not{p}_2\gamma_\mu\not{p}_{2'}\gamma_\nu\not{p}_1] \tag{3.36}$$

FeynCal?

3.4.2 $qg \to qg$ Compton scattering

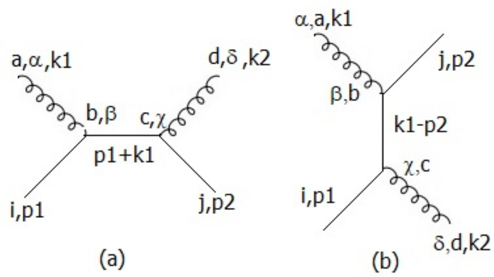


Figure 3.4: $qg \to qg$ Compton scattering diagrams.

3.4.3 $gg \to q\bar{q}$ pair creation

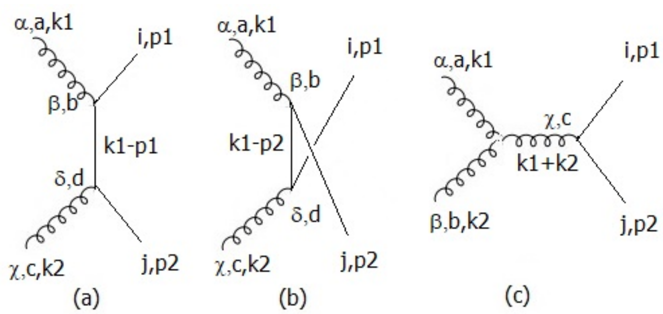


Figure 3.5: $gg \to q\bar{q}$ pair creation diagrams.