6 Effective Action

We do the formulation real scalar field function $\phi(x)$, with the field action functional $S[\phi]$, application to other field can be done easily.

6.1 Getting the effective action

We start from the generating functional

$$Z[j] = \int \mathcal{D}[\phi] e^{iS[\phi]+i\int d^4 j(x)\phi(x)} \tag{6.1}$$

which is used to generate n-point correlation function

$$\begin{aligned} C(x_1,...,x_n) = \langle \phi(x_1)...\phi(x_n) \rangle &= \frac{\int \mathcal{D}[\phi]\phi(x_1)...\phi(x_n)e^{iS[\phi]}}{\int \mathcal{D}[\phi]e^{iS[\phi]}} \\ &= \left[\frac{(-i)^n}{Z[j]} \frac{\delta^n Z[j]}{\delta j(x_1)...\delta j(x_n)} \right]_{j=0} \end{aligned} \tag{6.2}$$

In diagram representation $C(x_1,...,x_n)$ contains all connected and disconnected diagrams. However we can define the generating functional that generate only the connected diagram as

$$W[j] = -i \ln Z[j] \rightarrow Z[j] = e^{iW[j]} \tag{6.3}$$

Let us determine

$$\frac{\delta W[j]}{\delta j(x)} = \frac{-i}{Z[j]} \frac{\delta Z[j]}{\delta j(x)} = \langle \phi(x) \rangle_j \tag{6.4}$$

$$\begin{aligned} \frac{\delta^2 W[j]}{\delta j(x)\delta j(y)} &= \left(\frac{(-i)^2}{Z[j]} \frac{\delta^2 Z[j]}{\delta j(x)\delta j(y)} - \frac{1}{Z^2[j]} \frac{\delta Z[j]}{\delta j(x)} \frac{\delta Z[j]}{\delta j(y)} \right) \\ &= \langle \phi(x)\phi(y) \rangle_j - \langle \phi(x) \rangle_j \langle \phi(y) \rangle_j \\ &= \langle \phi(x)\phi(y) \rangle_{j,connected} \end{aligned} \tag{6.5}$$

As we observe from (6.4) $\langle \phi(x) \rangle$ is a classical value, and let us define

$$\Phi(x) = \langle \phi(x) \rangle = \frac{\delta W[j]}{\delta j(x)} \tag{6.6}$$

We can see that $\Phi(x)$ is a functional of $j(x)$. Practically we can apply Legendre transformation of $W[j]$ into a new functional of $\Phi(x)$ as

$$\Gamma[\Phi] = W[j] - \int d^4 j(x)\Phi(x) \tag{6.7}$$

It is called *effective action*. Let us determine

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = \int d^4y \frac{\delta W[j]}{\delta j(y)} \frac{\delta j(y)}{\delta\Phi(x)} - j(x) - \int d^4y \frac{\delta j(y)}{\delta\Phi(x)}\Phi(y) = -j(x) \tag{6.8}$$

Therefore the effective action $\Gamma[\Phi]$ is the quantity that generates the quantum corrected classical equation of motion by being extremized with respect to variation of the vacuum expectation value of the field.

By inversion, we can recover $W[j]$ in the form

$$W[j] = \Gamma[\Phi] + \int d^4x j(x)\Phi(x) \tag{6.9}$$

So that we can have the relation between effective action $\Gamma[\Phi]$ and action function $S[\phi_{cl}]$ at classical level as

$$Z[j] = e^{iW[j]} = e^{i\Gamma[\Phi]+i\int d^4x j(x)\Phi(x)} = \int \mathcal{D}[\phi] e^{iS[\phi]+i\int d^4x j(x)\phi(x)} \tag{6.10}$$

$$\text{with the classical EOM } \left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_{cl}} = -j(x) \tag{6.11}$$

Therefore one can say that

$$W[j] = \Gamma[\Phi] + \int d^4x j(x)\Phi(x) = S[\phi_{cl}] + \int d^4x j(x)\phi_{cl}(x) \tag{6.12}$$

Apply with functional derivative, we will have

$$\frac{\delta W[j]}{\delta j(x)} = \Phi(x) = \int d^4y \frac{\delta S[\phi_{cl}]}{\delta\phi_{cl}(y)} \frac{\delta\phi_{cl}(y)}{\delta j(x)} + \phi_{cl}(x) + \int d^4y j(y) \frac{d\phi_{cl}(y)}{\delta j(x)} \tag{6.13}$$

$$\rightarrow \Phi(x) = \phi_{cl}(x) \tag{6.14}$$

At the quantum level, we also have the equality

$$\Gamma[\Phi] = S[\phi] \tag{6.15}$$

6.2 Effective potential

Generally, one can write the functional form of the effective action in the form

$$\Gamma[\Phi] = \sum_{n=0}^{\infty} \int ... \int \Gamma^{(n)}(x_1, ..., x_n)\Phi(x_1)...\Phi(x_n) \tag{6.16}$$

where

$$\Gamma^{(n)}(x_1, ..., x_n) = \left. \frac{\delta^n \Gamma[\Phi]}{\delta\Phi(x_1)...\delta\Phi(x_n)} \right|_{\Phi=\phi_{cl}} \tag{6.17}$$

For example case of the real scalar field action, with minimal potential appear at $\phi = 0$, we have

$$S[\phi_{cl}] = \int d^4x \left(\frac{1}{2}(\partial\phi(x) - \frac{1}{2}m^2\phi^2(x) - \mathcal{V}(\phi) \right) = \Gamma[\Phi] \tag{6.18}$$

$$\to \Gamma^{(2)}(x_1, x_2) = (-\partial^2 - m^2)\delta^{(4)}(x_1 - x_2) \equiv G_0^{-1}(x_1, x_2) \tag{6.19}$$

$$\Gamma^{(4)} = \frac{\lambda}{3!} \tag{6.20}$$

We now look for the quantum corrections within the effective action, by doing Taylor's expansion of the quantum field around its classical value with some quantum fluctuation as

$$\phi(x) = \phi_{cl}(x) + \eta(x) \tag{6.21}$$

So that

$$\begin{aligned} S[\phi] + \int d^4 j(x)\phi(x) &= S[\phi_{cl} + \eta] + \int d^4xj(x)(\phi_{cl}(x) + \eta(x)) \\ &= S[\phi_{cl}] + \frac{1}{2}d^4x \int d^4yS''[\phi_{cl}]\eta(x)\eta(y) + ... \\ &\quad + \int d^4xj(x)\phi_{cl}(x) + \int d^4xj(x)\eta(x) \end{aligned} \tag{6.22}$$

$$\to Z[j] \simeq e^{iS[\phi_{cl}]+i\int d^4xj(x)\phi_{cl}(x)} \int \mathcal{D}[\eta]e^{\frac{i}{2}\int d^4x \int d^4yS''[\phi_{cl}]\eta(x)\eta(y)} \tag{6.23}$$

$$\simeq e^{iS[\phi_{cl}]+i\int d^4xj(x)\phi_{cl}(x)} \left(\det |S''[\phi_{cl}]|\right)^{-1/2} \tag{6.24}$$

Using identity

$$(\det A)^{-1/2} = e^{-\frac{1}{2}Tr\ln A}$$

From (6.15), we will observe that

$$\Gamma[\Phi] = S[\phi_{cl}] + \frac{i}{2}Tr\ln S''[\phi_{cl}] \tag{6.25}$$

Now let determine $S''[\phi_{cl}]$, from (6.18) we have

$$S''[\phi_{cl}] \;=\; (-\partial^2 - m^2) - \mathcal{V}''(\phi_{cl}^2) \tag{6.26}$$

From (6.25), we will have

$$\Gamma[\Phi] = S[\phi_{cl}] + \frac{i}{2}Tr\ln(-\partial^2 - m^2 - \mathcal{V}''(\phi_{cl})) \tag{6.27}$$

$$\begin{aligned} \rightarrow Tr\ln(-\partial^2 - m^2 - \mathcal{V}'') &= \int d^4x \ln(-\partial_x^2 - m^2 - \mathcal{V}''(\phi_{cl})) \\ &= \int d^4x \int \frac{d^4k}{(2\pi)^4}\ln(k^2 - m^2 - \mathcal{V}''(\phi_{cl})) \end{aligned} \tag{6.28}$$

We can define the effective potential in the form

$$\mathcal{V}_{eff}(\phi_{cl}) = \mathcal{V}(\phi_{cl}) - \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2 - m^2 - \mathcal{V}''(\phi_{cl})) \tag{6.29}$$

6.3 Loop expansion

Let us determine

$$\begin{aligned} -\frac{i}{2}Tr\ln(-\partial^2 - m^2 - \mathcal{V}'') &= -\frac{i}{2}Tr\ln\left[(-\partial^2 - m^2)\left(1 - \frac{\mathcal{V}''}{-\partial^2 - m^2}\right)\right] \\ &= -\frac{i}{2}Tr\ln(-\partial^2 - m^2) - \frac{i}{2}Tr\ln\left(1 - \frac{\mathcal{V}''}{-\partial^2 - m^2}\right) \end{aligned} \tag{6.30}$$

The first part is the usual *Coleman-Weinberg potential*. For the second part, let us determine

$$\frac{i}{2}Tr\ln\left(1 + \frac{i}{-\partial^2 - m^2}i\mathcal{V}''\right)$$

$$= \frac{i}{2}\int d^4x \int d^4y \ln\left(1 + \frac{i}{-\partial^2 - m^2}i\mathcal{V}''\right)(x,y)\delta^{(4)}(x,y) \tag{6.31}$$

$$= \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}\int d^4x \int d^4y \left(\frac{i}{-\partial^2 - m^2}i\mathcal{V}''\right)^n (x,y)\delta^{(4)}(x,y) \tag{6.32}$$

Note that $\Delta(x,y) = \frac{i}{-\partial^2 - m^2}(x,y)$, for $n=1$ term we have

$$\frac{i}{2}\int d^4x \int d^4y \Delta(x,y)\mathcal{V}''(\phi_{cl}(x))\delta^{(4)}(x,y) \tag{6.33}$$

Similarly, for $n=2$ term we have

$$\frac{i}{2\cdot 2}\int d^4x \int d^4y \int d^4z \delta^{(4)}(x,z)\Delta(x,y)\mathcal{V}''(\phi_{cl}(y))\Delta(y,z)\mathcal{V}''(\phi_{cl}(z)) \tag{6.34}$$

For $n=3$ term we have

$$\begin{aligned} \frac{i}{2\cdot 2\cdot 3}\int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 \delta^{(4)}(x_1,x_4)\Delta(x_1,x_2)\mathcal{V}''(\phi_{cl}(x_2)) \\ \times\Delta(x_2,x_3)\mathcal{V}''(\phi_{cl}(x_3))\Delta(x_3,x_4)\mathcal{V}''(\phi_{cl}(x_4)) \end{aligned} \tag{6.35}$$

For scalar ϕ^4-interacting model

$$\mathcal{V}(\phi) = \frac{\lambda}{4!}\phi^4(x) \to \mathcal{V}'' = \frac{\lambda}{2}\phi^2$$

The corresponding diagrams of the loop expansion of the effective potential are

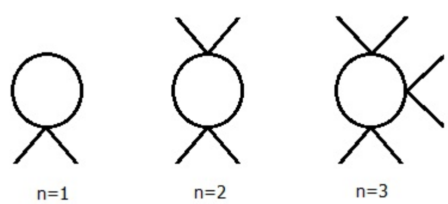

Figure 6.1:

6.4 Heat kernel

In order to calculate Coleman-Weinberg potential.