7 Effective Field Theory

7.1 Basic ideas

In order to determine physics at some energy scale, we need not to know the physical theory for all energy scale. The effective field theory (EFT) is constructed for this purpose. Inversely, from the known physics at some energy scale, we can construct a physical theory for all energy scale.

There are two EFT construction schemes:

- top-down construction, from the known theory
- bottom-up construction, from the unknown theory

7.2 Top-down construction of EFT

The original construction of EFT is to consider the low energy limit of the known field theory. For the example of scalar field ϕ, with the Lagrangian $\mathcal{L}(\phi)$, the field is separated into the light and heavy modes at the cutoff scale Λ as

$$\phi = \phi_L + \phi_H \to \mathcal{L} = \mathcal{L}(\phi_L, \phi_H) \tag{7.1}$$

Within the functional integral formulation, the*Wilson's effective action* is derived by integrating out the heavy mode

$$e^{iS_W[\phi_L]} = \int \mathcal{D}[\phi_H] e^{iS[\phi_L, \phi_H]} \tag{7.2}$$

Actually this integral is hard to perform, i.e, it can be done with diagramatic perturbation theory or saddle-point (Semi-classical) approximation.

Modern construction can be done by using the fact that after integrating out the heavy field, it left out with local operators of light field. So that the effective Lagrangian can be constructed in term of the series of local operators of light field as

$$\mathcal{L}_{eff}(\phi) = \mathcal{L}(\phi) + \sum_i g_i \mathcal{O}_i(\phi) \tag{7.3}$$

where we have denoted ϕ for the light field mode. The possible values of the coefficients g_i and the local operator $\mathcal{O}_i(\phi)$ are determined by the processes of (a) *power counting* and (b) *matching*.

7.2.1 Power counting

The power counting can be done via i) dimensional analysis or ii) scaling .

To see how it can be constructed, let us see an example of scalar field theory

$$\mathcal{L}(\phi,\Phi) = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m\phi^2 - \frac{1}{2}\partial\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 - V(\phi,\Phi) \tag{7.4}$$

$$\rightarrow \mathcal{L}_{eff}(\phi) = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m\phi^2$$
$$+A(\phi) + B(\phi)(\partial_\mu\phi\partial^\mu\phi) + C(\phi)(\partial_\mu\phi\partial^\mu\phi)^2 + ... \tag{7.5}$$

with $m < \Lambda < M$, and

$$A(\phi) = a_2\phi^4 + a_3\phi^6 + ... \tag{7.6}$$
$$B(\phi) = b_1\phi^2 + b_2\phi^4 + ... \tag{7.7}$$

where even order of ϕ is selected according to the even symmetry $\phi \rightarrow -\phi$ of the Lagrangian.

i) Dimensional analysis: In natural unit $c = 1 = h$, we will observe that $L = T = M^{-1} = E^{-1}$, and the action is dimensionless. So that in D-dimensional spacetime,

$$[d^Dx] = -D,\ [\mathcal{L}] = D,\ [\partial_\mu] = 1 \tag{7.8}$$
$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi \rightarrow [\phi] = D/2 - 1 \tag{7.9}$$
$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi \rightarrow [\psi] = (D-1)/2 \tag{7.10}$$
$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\mu(\partial^\mu A^\nu - \partial^\mu A^\nu) \rightarrow [A_\mu] = D/2 - 1 \tag{7.11}$$

Suppose that

$$[g_i] = \delta_i \rightarrow [\mathcal{O}_i] = D - \delta_i \tag{7.12}$$

Let Λ be the cutoff, let us define the dimensionless coefficients

$$\lambda_i = \frac{g_i}{\Lambda^{D-\delta_i}} \rightarrow I = \int d^Dx g_i\mathcal{O}_i \sim \lambda_i\left(\frac{E}{\Lambda}\right)^{\delta_i - D} \tag{7.13}$$

As we determine this result in the limit $E \rightarrow 0$, we will observe that

dimension	as $E \to 0$	name
$\delta_i < D$	I-grows	relevant
$\delta_i = D$	I-constant	marginal
$\delta_i > D$	I-shrinks	irrelevant

ii) Scaling: Rewrite the effective Lagrangian in the form

$$\mathcal{L}_{eff}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \sum_n \left(\frac{a_n}{\Lambda^{2n}}\phi^{4+2n} + \frac{b_n}{\Lambda^{2n}}\phi^{2+2n}(\partial\phi)^2 + ... \right) \tag{7.14}$$

Apply the scaling $\phi(x) \to \phi_s(x) = \phi(sx)$, this would yield

$$\begin{aligned} S_{eff}[\phi_s(x)] = \int d^4x \Bigg[&\frac{1}{2}(\partial_x\phi(sx))^2 + \frac{m^2}{2}\phi^2(sx) + \frac{\lambda}{4!}\phi^4(sx) \\ ... + &\sum_n \left(\frac{a_n}{\Lambda^{2n}}\phi^{4+2n}(sx) + \frac{b_n}{\Lambda^{2n}}\phi^{2+2n}(sx)(\partial_x\phi(sx))^2 + ... \right) \Bigg] \end{aligned} \tag{7.15}$$

Now define a new variable $y = sx$, then $d^4x = s^{-4}d^4y, \partial_x = s\partial_y$. Also define $\tilde{\phi}(y) = s^{-1}\phi(y)$, then we have

$$\begin{aligned} S_{eff}[\tilde{\phi}(y)] = \int d^4y \Bigg[&\frac{1}{2}(\partial_y\tilde{\phi}(y))^2 + \frac{m^2s^{-2}}{2}\tilde{\phi}^2(y) + \frac{\lambda}{4!}\tilde{\phi}^4(y) \\ ... + &\sum_n \left(\frac{a_n s^{2n}}{\Lambda^{2n}}\tilde{\phi}^{4+2n}(y) + \frac{b_n s^{2n}}{\Lambda^{2n}}\tilde{\phi}^{2+2n}(y)(\partial_y\tilde{\phi}(y))^2 + ... \right) \Bigg] \end{aligned} \tag{7.16}$$

From this scaling scheme, we observe that

$$\phi \to s^{-1}\phi \tag{7.17}$$
$$m^2 \to s^{-2}m^2 \tag{7.18}$$
$$\lambda \to \lambda \tag{7.19}$$
$$a_n \to s^{2n}a_n \tag{7.20}$$
$$b_n \to s^{2n}b_n \tag{7.21}$$

Their behaviors in the limit $s \to 0$ are classified to be

quantity	as $s \to 0$	name
m^2	grows	relevant
λ	constant	marginal
a_n, b_n	falls	irrelevant

iii) Naturalness: There is one further constraint of effective theories, beyond symmetry and locality of operators we have determined in power counting, is naturalness. Let us determine the mass term of the scalar field, in which $[m^2] = 2$ for $D = 4$. Let us rewrite in term of the dimensionless coupling $\lambda = m^2/\Lambda^2$, so that

$$\frac{1}{2}m^2\phi^2 = \frac{1}{2}(\lambda\Lambda^2)\phi^2 \tag{7.22}$$

which appear at the cutoff energy. To be natural, this term should not contain the effective field theory.

7.2.2 Tree-level matching

We can understand matching by example. Let us determine complex scalar field with SSB potential

$$\mathcal{L} = -\partial_\mu\phi^*\partial^\mu\phi - \frac{\lambda^2}{4}\left(\phi^*\phi - v^2\right)^2 \tag{7.23}$$

This Lagrangian has global U(1) symmetry $\phi \to e^{i\alpha}\phi$. Let us redefine

$$\phi(x) = \chi(x)e^{i\theta(x)} \tag{7.24}$$

$$\to \mathcal{L} = -\partial_\mu\chi\partial^\mu\chi - \chi^2\partial_\mu\theta\partial^\mu\theta - \frac{\lambda^2}{4}\left(\chi^2 - v^2\right) \tag{7.25}$$

We observe that θ is massless, and χ is massive, with mass $M = \lambda v$. SSB is defined by writing

$$\chi = v + \frac{1}{\sqrt{2}}\psi,\ \theta = \frac{1}{\sqrt{2}}\xi$$

Then we have

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\psi\right)^2\partial_\mu\xi\partial^\mu\xi - \frac{\lambda^2}{4}\left(\sqrt{2}v\psi + \frac{1}{2}\psi^2\right)^2 \tag{7.26}$$

This Lagrangian contains four types of vertices, see figure (7.1).

Let us determine the amplitude of $\xi\xi \to \xi\xi$ scattering, with the the conserved momenta $p + q = p' + q'$.

We will have

$$\mathcal{M} = \frac{2}{v}\left[\frac{(p\cdot q)^2}{(p+q)^2 + M^2} + \frac{(p\cdot p')^2}{(p-p')^2 + M^2} + \frac{(p\cdot q')^2}{(p-q)^2 + M^2}\right] \tag{7.27}$$

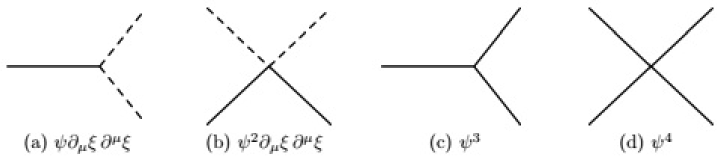

Figure 7.1:

Let us determine

$$\frac{1}{X^2+M^2}=\frac{1}{M^2}\left[1-\frac{X^2}{M^2}+\frac{X^4}{M^4}+...\right] \tag{7.28}$$

So that

$$\mathcal{M}=\frac{2}{v^2M^2}\left[(p\cdot q)^2+(p\cdot p')^2+(p\cdot q')^2\right]+O\left(\frac{1}{M^2}\right) \tag{7.29}$$

Now we have to construct the effective Lagrangian of ξ field that correspond to the amplitude of $\xi\xi\to\xi\xi$ above, it is

$$\mathcal{L}_{eff}(\xi)=-\frac{1}{2}\partial_\mu\xi\partial^\mu\xi-a(\partial_\mu\xi\partial^\mu\xi)^2 \tag{7.30}$$

$$\to a=\frac{1}{4v^2M^2} \tag{7.31}$$

See figure (7.2)

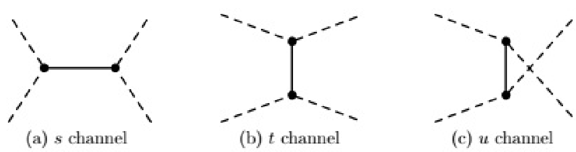

Figure 7.2:

Example 1: Rayleigh scattering (blue sky)

Rayleigh scattering = elastic scattering of light from atoms, will be determined as effective field theory of low energy QED. The zeroth order Lagrangian is

$$\mathcal{L}^{(0)}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+i\phi^*v^\mu\partial_\mu\phi \tag{7.32}$$

where ϕ is non-relativistic atomic field and $v^\mu = (1,0,0,0)$ is 4-velocity, satisfy a condition $v_\mu v^\mu = 1$. With dimensional analysis suggest that the the first order $d=6$ effective Lagrangian should be in the form

$$\mathcal{L}^{(1)} = a_0^3 c_a \phi^* \phi F_{\mu\nu} F^{\mu\nu} + a_0^3 c_2 \phi^* \phi v_\mu v^\nu F^{\mu\alpha} F_{\nu\alpha} \tag{7.33}$$

The tree-level elastic scattering amplitude will be

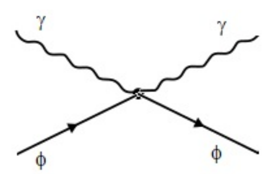

Figure 7.3:

$$\mathcal{M}(\gamma\phi \to \gamma\phi) \propto a_0^3 E_\gamma^2 \tag{7.34}$$

$$\sigma(\gamma\phi \to \gamma\phi) \propto a_0^6 E_\gamma^4 \tag{7.35}$$

$$\frac{\sigma(\gamma\phi \to \gamma\phi)(blue)}{\sigma(\gamma\phi \to \gamma\phi)(red)} = \frac{E_\gamma^4(blue)}{E_\gamma^4(red)} = \frac{\lambda^4(red)}{\lambda^4(blue)} = 8 \tag{7.36}$$

That is why the sky is blue.

Example 2: light-by-light scattering

In regular QED we can not have light-by-light scattering at tree level, but the one-loop level. At the low energy QED, $E_\gamma \ll m_e$, we can have the

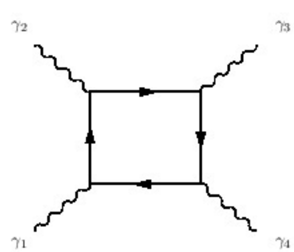

Figure 7.4:

effective field theory of light-by-light scattering at tree level. The zero order Lagrangian is

$$\mathcal{L}^{(0)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{7.37}$$

As we have observe that $[F_{\mu\nu}] = 2$, from dimensional analysis suggest that the first order effective Lagrangian should be in the form

$$\mathcal{L}^{(1)} = \frac{c_1}{32m_e^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{c_2}{16m_e^4}F_{\mu\rho}F^{\nu\rho}F^{\mu\sigma}F_{\nu\sigma} \tag{7.38}$$

where c_1, c_2 are dimensionless unknown coefficients. Note that $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ is called *Euler-Heisenberg Lagrangian* The elastic scattering amplitude at tree level is

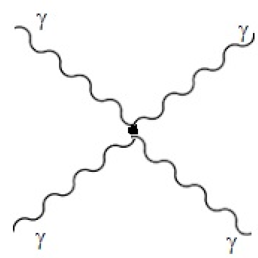

Figure 7.5:

$$\mathcal{M}(\gamma\gamma \to \gamma\gamma) \propto \frac{E_\gamma^4}{m_e^4} \tag{7.39}$$

$$\sigma(\gamma\gamma \to \gamma\gamma) \propto \frac{E_\gamma^8}{m_e^8} \tag{7.40}$$

We observe that it is suppressed by a factor of m_e^8.

Example 3: Fermi's theory of weak interaction

Let us determine the weak decay of muon, see figure (6),

$$\mu^- \to \nu_\mu + e^- + \bar{\nu}_e$$

The effective Lagrangian is

$$\mathcal{L}_{eff} \supset \frac{c}{\Lambda^2}(\bar{\nu}_\mu \bar{\sigma}^\rho \mu)(\bar{e}\bar{\sigma}_\rho \nu_e) + h.c. \tag{7.41}$$

Form dimensional analysis, this shows that $[\nu_\mu] = [\mu] = [e] = [\nu_e] = \frac{3}{2} \to [\Lambda] = 1$, while c is dimensionless coefficient. The tree level amplitude of this decay is

$$\mathcal{M} = \frac{c}{\Lambda^2}\bar{U}(k_{\nu_\mu})\bar{\sigma}^\rho U(k_\mu)\bar{U}(k_e)\bar{\sigma}_\rho V(k_{\nu_e}) \tag{7.42}$$

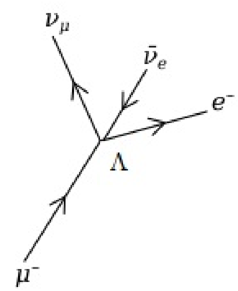

Figure 7.6:

The decay rate is

$$\frac{d\Gamma(\mu \to \nu_\mu e \bar{\nu}_e)}{dq^2} = \frac{c^2(m_\mu^2 - q^2)^2(m_\mu^2 + 2q^2)}{768\pi^3 m_\mu^3 \Lambda^4} \tag{7.43}$$

Matching with the standard model weak interaction at leading order, see figure (7), we observe that

$$\frac{c}{\Lambda^2} = 2\sqrt{2}G_F = \frac{1}{(174GeV)^2}$$

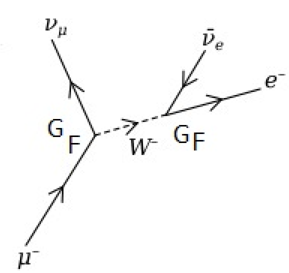

Figure 7.7:

7.2.3 Loop-order matching

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7.3 Bottom-up construction of EFT

In this case we extend the known theory into the energy range that not cover by the known theory, i.e., beyond standard model (BSM) extension of the standard model (SM) theory. The SM theory is work well below the electroweak (EW) scale, so that the BSM theory will be its UV extension. The effective field theory Lagrangina of the BSM will be written in term of the SM model Lagrangian plus the series of local operators of the SM fields, as

$$\mathcal{L}_{BSM} = \mathcal{L}_{SM}(\phi_{SM}) + \sum_i g_i \mathcal{O}(\phi_{SM}) \tag{7.44}$$