8 Chiral Symmetry and Chiral Field Theory

8.1 Chiral symmetry

Let us determine massless Dirac Lagrangian

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi \tag{1}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ is Feynman slash, and γ^μ is Dirac gamma matrix satisfy Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Note that ψ is a four component Dirac spinor, satisfy masssless Dirac equation $i\partial\!\!\!/\psi = 0$. For Dirac representation of the gamma matrix

$$\gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \ i = 1, 2, 3 \tag{2}$$

where 1_2 is 2x2 identity matrix and $\{\sigma^i\}$ is a set of Pauli's matrices. We will observe that the positive energy solution $\psi^+(x) \sim U(k)e^{-ik\cdot x}$ and negative energy solution $\psi^-(x) \sim V(k)e^{ik\cdot x}$ satisfy the same equation. In order to make them different, we have to use Weyl representation of the gamma matrix, as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \text{ with } \sigma^\mu = (1, \vec{\sigma}), \ \bar{\sigma}^\mu = (1, -\vec{\sigma}) \tag{3}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

The Dirac spinor can be written in terms of two Weyl spinors, as

$$U(k) = \begin{pmatrix} \bar{\chi} \\ \eta \end{pmatrix} \tag{5}$$

So that, the positive energy Dirac equation becomes

$$0 = k_\mu\gamma^\mu U = \begin{pmatrix} 0 & k_\mu\sigma^\mu \\ k_\mu\bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \bar{\chi} \\ \eta \end{pmatrix} \tag{6}$$

$$0 = k_\mu\sigma^\mu\eta = (k^0 - \vec{k}\cdot\vec{\sigma})\eta = k^0(1 - \hat{h})\eta, \rightarrow \hat{h}\eta = +1\eta \tag{7}$$

$$0 = k_\mu\bar{\sigma}^\mu\bar{\chi} = (k^0 + \vec{k}\cdot\vec{\sigma})\bar{\chi} = k^0(1 + \hat{h})\chi, \rightarrow \hat{h}\bar{\chi} = -1\bar{\chi} \tag{8}$$

with $k^\mu = (k^0, \vec{k})$, and $k^2 = 0 \rightarrow k^0 = |\vec{k}|$. And $\hat{h} = \frac{\vec{k}\cdot\vec{\sigma}}{|\vec{k}|}$, it is called *helicity operator* (handedness). From (6) η has helicity +1 or *right-handedness*, while χ has helicity -1 or *left-handedness*.

According to Dirac spinor ψ we can separate it into two helicity parts by using chiral operators

$$P_R = \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ P_L = \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{9}$$

$$\rightarrow P_R P_R = P_R, \ P_L P_L = P_L, \ P_R P_L = P_L P_R = 0 \tag{10}$$

So that

$$\psi_R = P_R\psi \sim \begin{pmatrix} 0 \\ \eta \end{pmatrix} e^{-ik\cdot x} \tag{11}$$

$$\psi_L = P_L\psi \sim \begin{pmatrix} \bar{\chi} \\ 0 \end{pmatrix} e^{-ik\cdot x} \tag{12}$$

$$\psi = \psi_R + \psi_L \tag{13}$$

And one can write the massless Dirac Lagrangian

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi = i\bar{\psi}_R\partial\!\!\!/\psi_R + i\bar{\psi}_L\partial\!\!\!/\psi_L \tag{14}$$

and the conserved vector current J^μ, i.e. $\partial_\mu J^\mu = 0$,

$$J^\mu = \bar{\psi}\gamma^\mu\psi = \bar{\psi}_R\gamma^\mu\psi_R + \bar{\psi}_L\gamma^\mu\psi_L = J^\mu_R + J^\mu_L \tag{15}$$

and the conserved axial vector current A^μ

$$J^\mu_5 = \bar{\psi}\gamma^\mu\gamma_5\psi = \bar{\psi}_R\gamma^\mu\psi_R - \bar{\psi}\gamma^\mu\psi_L = J^\mu_R - J^\mu_L \tag{16}$$

after we have used the fact that $\gamma_5\psi_{R/L} = \pm\psi_{R/L}$, and $\{\gamma^\mu, \gamma_5\} = 0$.

8.2 Axial symmetry

There are associated global U(1) symmetries

- $U_V(1)$ symmetry

$$\psi \to e^{-i\alpha}\psi = (1 - i\alpha)\psi,\ \bar{\psi} \to \bar{\psi}e^{i\alpha} = \bar{\psi}(1 + i\alpha) \tag{17}$$

$$J^\mu = \bar{\psi}\gamma^\mu\psi \to \bar{\psi}(1 + i\alpha)\gamma^\mu(1 - i\alpha)\psi \simeq \bar{\psi}\gamma^\mu\psi \tag{18}$$

- $U_A(1)$ symmetry

$$\psi \to e^{-i\alpha\gamma_5}\psi = (1 - i\alpha\gamma_5)\psi,\ \bar{\psi} \to \bar{\psi}e^{-i\alpha\gamma_5} = \bar{\psi}(1 - i\alpha\gamma_5) \tag{19}$$

$$J^\mu_5 = \bar{\psi}\gamma^\mu\gamma_5\psi \to \bar{\psi}(1 - i\alpha\gamma_5)\gamma^\mu\gamma_5(1 - i\alpha\gamma_5)\psi \simeq \bar{\psi}\gamma^\mu\gamma_5\psi \tag{20}$$

Let us determine the mass term

$$\mathcal{L}_m = -m\bar{\psi}\psi \tag{21}$$

We can observe that it is invariant under $U_V(1)$ transformation but non invariant under $U_A(1)$ transformation. So that this term brakes chiral symmetry of the model Lagrangian.

8.3 Chiral symmetry of QCD

Let us determine the QCD Lagrangian with the $U(1) \times SU(2_f) \times SU(3_c)$ gauge symmetry, with the quark flavor spinor

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \tag{22}$$

Its Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\not{D} - M)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} \tag{23}$$

$$D_\mu = \partial_\mu - iq_c A_\mu,\ F_{\mu\nu} = \frac{i}{q_c}[D_\mu, D_\nu] \tag{24}$$

$$A_\mu = A^a_\mu t^a,\ [t^a, t^b] = if^{abc}t^c \to su(3_c) - algebra \tag{25}$$

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \tag{26}$$

8.4 Chiral effective field theory