# 2 Supersymmetry Algebra and Representations

## 2.1 Weyl spinors and spinor indices

Massless Dirac equation

$$i\partial\!\!\!/\psi(x) = 0 \tag{2.1}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$, and $\gamma^\mu$ is Dirac gamma matrix. We cannot solve this equation within Dirac representation of $\gamma^\mu$, since we cannot make the different between the positive energy $U(k)$ spinor and negative energy $V(k)$ spinor. Anyway, we can have different solutions for $U$ and $V$ spinors when using Weyl representation of $\gamma^\mu$, i.e.,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \ \sigma^\mu = (1, \vec{\sigma}), \ \bar{\sigma}^\mu = (1, -\vec{\sigma}) \tag{2.2}$$

For positive energy solution, $\psi^{(+)}(x) \sim U(k)e^{-ik\cdot x}$, then we have (2.1)

$$k_\mu \gamma^\mu U(k) = 0 \tag{2.3}$$

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & k_\mu \sigma^\mu \\ k_\mu \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \tag{2.4}$$

$$(k^0 - \vec{k}\cdot\vec{\sigma})u_2(k) = 0 \rightarrow k^0(1 - \hat{h})u_2(k) = 0 \tag{2.5}$$

$$(k^0 + \vec{k}\cdot\vec{\sigma})u_1(k) = 0 \rightarrow k^0(1 + \hat{h})u_(k) = 0 \tag{2.6}$$

where $\hat{h} = \frac{\vec{k}\cdot\vec{\sigma}}{|\vec{k}|}$ is helicity operator, with $k^2 = 0$. (2.5) shows that $u_2$ is right handedness spinor ($\hat{h} = +1$), while (2.6) shows that $u_1$ is left handedness spinor ($\hat{h} = -1$). let us denote

$$u_2 = \chi_\alpha, \ \alpha = 1, 2 \qquad \text{right handedness spinor basis} \tag{2.7}$$

$$u_1 = \bar{\eta}^{\dot{\alpha}}, \ \ \dot{\alpha} = \dot{1}, \dot{2} \qquad \text{left handedness spinor basis} \tag{2.8}$$

$$\psi_D^{(+)} \sim \begin{pmatrix} \bar{\eta}^{\dot{\alpha}} \\ \chi_\alpha \end{pmatrix} e^{-ik\dot{x}} \tag{2.9}$$

Note that $\alpha, \beta, ... = 1, 2$ and $\dot{\alpha}, \dot{\beta}, ... = \dot{1}\dot{2}$ are called spinor indices.

For the negative energy solution, $\psi^{(-)}(x) \sim V(k)e^{ik\cdot x}$, we will have

$$\psi_D^{(-)} \sim \begin{pmatrix} \chi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix} e^{ik\cdot x} \tag{2.10}$$

In order to get a full spinor expression of Dirac equation, we will also index the gamma matrix as in the following

$$(2.2) \rightarrow \sigma^\mu = (\sigma^\mu)^{\dot{\alpha}\beta}, \qquad \bar{\sigma}^\mu = (\bar{\sigma}^\mu)_{\alpha\dot{\beta}} \tag{2.11}$$

$$(2.3) \rightarrow k_\mu(\sigma^\mu)^{\dot{\alpha}\beta}\chi_\beta = 0, \qquad k_\mu(\bar{\sigma}^\mu)_{\alpha\dot{\beta}}\bar{\chi}^{\dot{\beta}} = 0 \tag{2.12}$$

When we work with spinors and fermionic opertors, assigned with spinor indices, we will obey the following rule

$$(\chi_\alpha)^\dagger = \bar{\chi}_{\dot{\alpha}}, \qquad (\bar{\chi}^{\dot{\alpha}})^\dagger = \chi^\alpha \tag{2.13}$$

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta,\ \chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta, \qquad \bar{\chi}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}},\ \bar{\chi}_{\dot{\alpha}} = \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}} \tag{2.14}$$

$$1 = \epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12}, \qquad 1 = \bar{\epsilon}^{\dot{1}\dot{2}} = -\bar{\epsilon}^{\dot{2}\dot{1}} = \bar{\epsilon}_{\dot{2}\dot{1}} = -\bar{\epsilon}_{\dot{1}\dot{2}} \tag{2.15}$$

Note that Majorana is real spinor, it expression in term of Weyl spinor will be

$$\psi_M \sim \begin{pmatrix} \bar{\chi}^{\dot{\alpha}} \\ \chi_\alpha \end{pmatrix} e^{-ik\cdot x} \tag{2.16}$$

## 2.2 Spinor representation of group

From a Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, its spinor representation is denote as

$$D[\Lambda_S] = \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right),\ S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \to\ (4x4) - matrix \tag{2.17}$$

Its action on Dirac spinor is

$$\psi'^a(x) = S[\Lambda]^a{}_b\psi^b(\Lambda^{-1}x), \quad a, b = 1, 2, 3, 4 \tag{2.18}$$

Its action on Weyl spinors are

$$\chi'_\alpha(x) = S[\Lambda]_\alpha{}^\beta\chi_\beta(\Lambda^{-1}x) \tag{2.19}$$

$$\bar{\chi}'^{\dot{\alpha}}(x) = S[\Lambda]^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}}(\Lambda^{-1}x) \tag{2.20}$$

where

$$S[\Lambda]_\alpha{}^\beta = \exp\left(\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta\right),\ (\sigma^{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\mu\bar{\sigma}^\mu)_\alpha{}^\beta \tag{2.21}$$

$$S[\Lambda]^{\dot{\alpha}}{}_{\dot{\beta}} = \exp\left(\frac{i}{2}\omega_{\mu\nu}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\right),\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} \tag{2.22}$$

## 2.3 Super-Poincare algebra

Poincare algebra, the algebra of bosonic operators $P_\mu$ and $M_{\mu\nu}$. The N-supersymmetry extension is done by introduction of the 2N-fermionic operators, written in form of Weyl spinor, as

$$Q^A = \begin{pmatrix} \bar{Q}^{A\dot{\alpha}} \\ Q^A_\alpha \end{pmatrix}, \quad A = 1, 2, ..., N \tag{2.23}$$

For convenient, the N=1 supersymmetric algebra are determined to be in the form

$$[Q_\alpha, P^\mu] = 0,\ [\bar{Q}_{\dot{\alpha}}, P^\mu] = 0 \tag{2.24}$$

$$[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \tag{2.25}$$

$$\left[\bar{Q}^{\dot{\alpha}}, M^{\mu\nu}\right] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{Q}^{\dot{\beta}} \tag{2.26}$$

$$\{Q_\alpha, Q_\beta\} = 0, \tag{2.27}$$

$$\{\bar{Q}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} = 0 \tag{2.28}$$

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\delta^{AB}(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu \tag{2.29}$$

These are called *N=1 super-Poincare algebra.*

The two Casimir operators of this algebra can be constructed from $P^\mu$ but not from $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$. Its extension with $Q$ can be done in the form of a tensor

$$C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu,\ \text{where } B_\mu = W_\mu + \frac{1}{2}X_\mu \tag{2.30}$$

$$\text{and } X_\mu = \frac{1}{2}\bar{Q}\gamma^\mu\gamma^5 Q \tag{2.31}$$

$$\rightarrow [C^2, Q] = 0,\ [C^2, P^\mu] = 0,\ [C^2, M^{\mu\nu}] = 0 \tag{2.32}$$

Let us determine the eigen-value/state of $C^2$, especially from the massive particle with $P^\mu = (m, 0, 0, 0)$ in its rest frame, we will have

$$\begin{aligned} C^2 &= 2B_\mu P_\nu B^\mu P^\nu - 2B_\mu P_\nu B^\nu P^\mu \\ &= 2m^2 B_\mu B^\mu - 2m^2 B_0^2 \\ &= 2m^2 B_k B^k \end{aligned} \tag{2.33}$$

$$\text{Since } B_k = W_k + \frac{1}{4}X_k = mS_k + \frac{1}{8}\bar{Q}\gamma_k\gamma^5 Q \equiv mJ_k \tag{2.34}$$

$$\rightarrow [J_i, J_j] = i\epsilon_{ijk}J_k,\ \text{so that } C^2 = 2m^4 J_k J^k \tag{2.35}$$

Note that $\vec{J}$ is called *superspin.* The representation of $C^2$ is determined from the eigen-value/state of $J^2$. Let us assign the eigen-state/value of $J^2$ as

$$J^2|j, j_3\rangle = j(j+1)|j, j_3\rangle,\quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ... \tag{2.36}$$

$$j_3 = -j, -(j-1), ..., (j-1), j \tag{2.37}$$

$$\text{Then}\ \ C^2|m; j, j_3\rangle = 2m^4 j(j+1)|m; j, j_3\rangle \tag{2.38}$$

$$\text{and simultaneously } P^2|m; j, j_3\rangle = m^2|m; j, j_3\rangle \tag{2.39}$$

## 2.4 N=1 supersymmetry representations

The representation is determined from the anti-commutation relation (2.29),

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu,$$

for the cases of massive and massless particle states.

### 2.4.1 Massless representation

In the light-front frame of massless particle of energy $E$, we have $p^\mu = (E, 0, 0, E)$. This results to $P^2 = 0$ and $C^2 = 0$. From (2,29), we will have

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2E(\sigma^0)_{\alpha\dot{\beta}} + 2E(\sigma^3)_{\alpha\dot{\beta}} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \tag{2.40}$$

$$\to \{Q_1, \bar{Q}_{\dot{1}}\} = 4E \tag{2.41}$$

$$\text{Define } a = \frac{Q_1}{\sqrt{4E}}, \quad a^\dagger = \frac{\bar{Q}_{\dot{1}}}{\sqrt{4E}} \to \{a, a^\dagger\} = 1 \tag{2.42}$$

From (2.34), we will observe that

$$[a, J^3] = \frac{1}{2}(\sigma^3)_{11} = \frac{1}{2}a \to aJ^3 - J^3 a = \frac{1}{2}a \tag{2.43}$$

$$\left[a^\dagger, J^3\right] = -\frac{1}{2}a^\dagger \tag{2.44}$$

$$J^3(a|E, \lambda >) = (J^3 a)|E, \lambda >= ((\lambda - \frac{1}{2})(a|E, \lambda >) \tag{2.45}$$

$$a|E, \lambda >\sim |E, \lambda - \frac{1}{2} > \tag{2.46}$$

$$\text{Similarly } a^\dagger|E, \lambda >\sim |E, \lambda + \frac{1}{2} > \tag{2.47}$$

Let us start to construct a set of *super-multiplets* by first assign a state of minimum helicity $|\Omega >$, where it is known as *Clifford vacuum* state, so that

$$|\Omega >= |E, \lambda >, a|\Omega >= 0, \; a^\dagger|\Omega >= |E, \lambda + 1/2 > \tag{2.48}$$

$$\text{Super}-\text{multiplets} \to |E, \pm\lambda >, |E, \pm(\lambda + 1/2) > \tag{2.49}$$

according PCT conjugate. For examples:

- $\lambda = 0$ is *chiral super-multiplete* $\to \{|E, 0 >, |E, \pm 1/2 >\}$
- $\lambda = 1/2$ is *vector/gauge super-multiplets* $\to \{|E, \pm 1/2 >, |E, \pm 1 >\}$
- $\lambda = 3/2$ is *gravity super-multiplets* $\to \{|E, \pm 3/2 >, |E, \pm 2 >\}$

| $\lambda = 0$ | $\lambda = 1/2$ |
|---|---|
| Higgs | Higgsion |
| selectron | electron |
| squark | quark |

| $\lambda = 1/2$ | $\lambda = 1$ |
|---|---|
| photino | photon |
| gluino | gluon |
| Wino, Zino | W,Z |

| $\lambda = 3/2$ | $\lambda = 2$ |
|---|---|
| gravitino | graviton |

### 2.4.2 Massive representation

In the rest frame of a particle with mass $m$, we have $p^\mu = (m, 0, 0, 0)$, then we have

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2m(\sigma^0)_{\alpha\dot{\beta}} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\beta}} \to \{Q_{1,2}, \bar{Q}_{\dot{1},\dot{2}}\} = 2m \tag{2.50}$$

$$\text{Define } a_{1,2} = \frac{Q_{1,2}}{\sqrt{2m}},\ a^\dagger_{1,2} = \frac{\bar{Q}_{\dot{1},\dot{2}}}{\sqrt{2m}} \to \{a_{1,2}, a^\dagger_{1,2}\} = 1 \tag{2.51}$$

Now let us define the Clifford vacuum state $|\Omega>=|m; j, j_3>$, and $a_{1,2}|\Omega>=0$. Note that

$$J_3|\Omega>= S_3|\Omega + \frac{1}{4}\bar{Q}\bar{\sigma}_i Q|\Omega>= S_3|\Omega>\to j_3 = s_3 = 1/2 \tag{2.52}$$

This means that the SUSY operators when act on state with superspin $j = y$ will result to states with superspin $j = y \pm 1/2$. Similar to (2.43, 44), we observe that

$$\left[Q, J^3\right] = \frac{1}{2}\sigma^3 Q \to \left[a_1, J^3\right] = \frac{1}{2}a_1,\ \left[a_2, J^3\right] = -\frac{1}{2}a_2 \tag{2.53}$$

$$\left[\bar{Q}, J^2\right] = -\frac{1}{2}\sigma^3\bar{Q} \to \left[a^\dagger_1, J^2\right] = -\frac{1}{2}a^\dagger_1,\ \left[a^\dagger_2, J^2\right] = \frac{1}{2}a^\dagger_2 \tag{2.54}$$

This means that $a_1, a_2^\dagger$ lower $j_3$ by 1/2, and $a_2, a_1^\dagger$ raise $j_3$ by 1/2.

Let us start with the Clifford vacuum $|\Omega>=|j, j_3>$, where the $m$ is hidden for convenient. The massive super-multiplet consist of

$$|\Omega>=|j = y, j_3>, \qquad 2j+1 \tag{2.55}$$

$$a_1^\dagger|j, j_3>, a_2^\dagger|j, j_3>\to|j \pm 1/2, j_3>, \qquad 2(j \pm 1/2) + 1 \tag{2.56}$$

$$a_1^\dagger a_2^\dagger|j, j_3>\to|j, j_3>, \qquad 2j+1 \tag{2.57}$$

For examples

- $j = 0$ *chiral super-multiplet*:

$$\phi, \phi', \psi_M$$

- $j = 1/2$ *vector super-multiplet*: 2 fermionic states (Dirac) + 1 scalar + 1 vector

$$\psi_D, \psi'_D, \phi, P_\mu$$

Note that in all super-multiplets, number of bosonic and fermionic degree of freedom are equal, i.e., $n_B = n_F$, the proof will placed later.