3 Supersymmetric Invariant Lagrangian

3.1 SUSY applied on fields

As we know from SUSY algebra that 2 SUSY transformations results to spacetime transformation

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \tag{3.1}$$

The we assigne SUSY transformation on field function of spactime in a similar fashion to spacetime translation, using unitary operation

$$U(a,\theta,\bar{\theta}) = e^{ia^\mu P_\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}} \tag{3.2}$$

$$\simeq 1 + ia^\mu P_\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}} + ... \tag{3.3}$$

where a^μ is infinitesimal spacetime translation, $\theta_\alpha, \dot{\theta}^{\dot{\alpha}}$ are infinitesimal spinorial parameters of SUSY transformation.

Let us first determine SUSY transformation on scalar field function $\phi(x)$, we will have

$$\phi(x) \to U(a,0,0)\phi(x)U^{-1}(a,0,0)$$

$$= \phi(x) - +ia^\mu[P_\mu, \phi(x)] + ... = \phi(x) + ia^\mu P_\mu \phi(x) + ... \tag{3.4}$$

$$\to \delta_a\phi(x) = ia^\mu P_\mu \phi(x) \tag{3.5}$$

$$\phi(x) \to U(0,\theta,\bar{\theta})\phi(x)U^{-1}(0,\theta,\bar{\theta})$$

$$= \phi(x) + i\theta^\alpha[Q_\alpha, \phi(x)] + i\bar{\theta}^{\dot{\alpha}}[\bar{Q}_{\dot{\alpha}}, \phi(x)] + ...$$

$$= \phi(x) + i\theta^\alpha Q_\alpha\phi(x) + i\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}\phi(x) + ... \tag{3.6}$$

$$\to \delta\phi(x) = i\theta^\alpha Q_\alpha \phi(x) + i\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}} = \delta_\theta\phi(x) + \delta_{\bar{\theta}}\phi(x) \tag{3.7}$$

The SUSY transformation on spinorial field function $\psi_\alpha(x)$ will be

$$\delta\psi_\alpha = i\theta^\beta Q_\beta\psi_\alpha + i\bar{\theta}^{\dot{\beta}}\bar{Q}_{\dot{\beta}}\psi_\alpha = \delta_\theta\psi_\alpha + \delta_{\bar{\theta}}\psi_\alpha \tag{3.8}$$

We can have similar expressions from $\delta\bar{\psi}^{\dot{\alpha}}$.

The possible value of $\delta\phi, \delta\psi_\alpha, \delta\bar{\chi}^{\dot{\alpha}}$ will be determine from their mass dimensions:

$$[P_\mu] = M, [\sigma^\mu] = 0 \to [Q] = [\bar{Q}] = M^{1/2}, [\theta] = [\bar{\theta}] = [M]^{-1/2} \tag{3.9}$$

$$[\phi(x)] = M,\ [\psi_\alpha] = [\bar{\psi}^{\dot{\alpha}}] = M^{3/2} \tag{3.10}$$

Since SUSY do transform bosonic field into fermionic field rand vice versa, from (3.7), we should have

$$\delta_\theta\phi(x) = \sqrt{2}\theta^\alpha\psi_\alpha(x) \tag{3.11}$$

$$\delta_\theta\psi_\alpha(x) = i\sqrt{2}\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\phi(x) \tag{3.12}$$

Let us determine from (3.7)

$$\begin{aligned}
[\delta_1,\delta_2]\,\phi(x) &= -\left[\theta_1 Q+\bar{\theta}_1\bar{Q},\theta_2 Q+\bar{\theta}_2\bar{Q}\right]\phi(x)\\
&= -\left(\theta_1\{Q,\bar{Q}\}\bar{\theta}_2-\theta_2\{Q,\bar{Q}\}\bar{\theta}_1\right)\phi(x)\\
&= -2\left(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1\right)P_\mu\phi(x)\\
&= 2i\left(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1\right)\partial_\mu\phi(x)
\end{aligned}\tag{3.13}$$

Similarly from (3.10), we will have

$$\begin{aligned}
[\delta_1,\delta_2]\,\phi(x) &= \delta_1\delta_2\phi(x)-\delta_2\delta_1\phi(x)\\
&= \sqrt{2}\theta_1\delta_2\psi(x)-\sqrt{2}\theta_2\delta_1\psi(x)\\
&= 2i\left(\theta_1\sigma^\mu\bar{\theta}_2-\theta_2\sigma^\mu\bar{\theta}_1\right)\partial_\mu\phi(x)
\end{aligned}\tag{3.14}$$

To be confident about the SUSY transformations (3.11) and (3.12), let us determine a similar algebra of (3.14) for spinorial field $\psi_\alpha(x)$, as

$$\begin{aligned}
[\delta_1,\delta_2]\,\psi_\alpha(x) &= \delta_1\delta_2\psi_\alpha(x)-\delta_2\delta_1\psi_\alpha(x)\\
&= i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}_1^{\dot{\alpha}}\partial_\mu\delta_2\phi(x)-i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}_2^{\dot{\alpha}}\partial_\mu\delta_1\phi(x)\\
&= 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}_1^{\dot{\alpha}}\theta_2^\beta\partial_\mu\psi_\beta(x)-2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}_2^{\dot{\alpha}}\theta_1^\beta\partial_\mu\psi_\beta(x)
\end{aligned}\tag{3.15}$$

In order to rearrange $\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}_1^{\dot{\alpha}}\theta_2^\beta\to\theta_2^\beta\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}_1^{\dot{\alpha}}$ we have to use what it is called *Fierz identities*, which is [1]

$$\bar{\chi}^{\dot{\alpha}}\eta^\alpha = -\frac{1}{2}(\eta^\beta\sigma_{\mu\beta\dot{\beta}}\bar{\chi}^{\dot{\beta}})\bar{\sigma}^{\mu\dot{\alpha}\alpha}\tag{3.16}$$

$$\to\bar{\theta}_1^{\dot{\alpha}}\theta_2^\beta = -\frac{1}{2}(\theta_2\sigma_\mu\bar{\theta}_1)\bar{\sigma}^{\mu\dot{\alpha}\beta},\ \bar{\theta}_2^{\dot{\alpha}}\theta_1^\beta = -\frac{1}{2}(\theta_1\sigma_\mu\bar{\theta}_2)\bar{\sigma}^{\mu\dot{\alpha}\beta}\tag{3.17}$$

Now (3.15) becomes

$$\begin{aligned}
[\delta_1,\delta_2]\,\psi_\alpha(x) &= i\sigma^\mu_{\alpha\dot{\alpha}}(\theta_1\sigma_\nu\bar{\theta}_2)\bar{\sigma}^{\nu\dot{\alpha}\beta}\partial_\mu\psi_\beta(x)\\
&\quad -i\sigma^\mu_{\alpha\dot{\alpha}}(\theta_2\sigma_\nu\bar{\theta}_1)\bar{\sigma}^{\nu\dot{\alpha}\beta}\partial_\mu\psi_\beta(x)\\
&= i\left((\theta_1\sigma_\nu\bar{\theta}_2)-(\theta_2\sigma_\nu\bar{\theta}_1)\right)\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta}\partial_\mu\psi_\beta(x)\\
&= i\left((\theta_1\sigma_\nu\bar{\theta}_2)-(\theta_2\sigma_\nu\bar{\theta}_1)\right)\left(2\eta^{\mu\nu}\delta_\alpha^\beta-\sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\right)\partial_\mu\psi_\beta(x)\\
&= 2i\left((\theta_1\sigma^\mu\bar{\theta}_2)-(\theta_2\sigma^\mu\bar{\theta}_1)\right)\partial_\mu\psi_\alpha\\
&\quad -i\left((\theta_1\sigma_\nu\bar{\theta}_2)-(\theta_2\sigma_\nu\bar{\theta}_1)\right)\sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu\psi_\beta(x)
\end{aligned}\tag{3.18}$$

The we have extra term for $[\delta_1,\delta_2]\psi_\alpha(x)$, when compared to (3.14). In order to have the same algebra, we have to introduce an extra term in $\delta\psi_\alpha(x)$ in (3.12), to make cancellation of the extra term appear in (3.18), as

$$\delta\psi_\alpha(x) = i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\phi(x)+\sqrt{2}\theta_\alpha f(x)\tag{3.19}$$

[1] See a proof in lecture note by R. Gurio.

where $f(x)$ is known as *auxiliary scalar field*. Of course we have to write its variation, using the fact that $[f] = M$, in the form

$$\delta f(x) = i\sqrt{2}\bar{\theta}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha(x) \tag{3.20}$$

Let us determine its contributions in the algebra of $\phi(x)$ and $\psi_\alpha(x)$ as

$$\begin{aligned}[\delta_1, \delta_2] &= \sqrt{2}\theta_1^\alpha\delta_2\psi_\alpha(x) - \sqrt{2}\theta_2^\alpha\delta_1\psi_\alpha(x) \\ &= ... + 2(\theta_1\theta_2)f - 2(\theta_2\theta_1)f = ... + 0\end{aligned} \tag{3.21}$$

$$\begin{aligned}[\delta_1, \delta_2]\,\psi_\alpha(x) &= \delta_1\delta_2\psi_\alpha(x) - \delta_2\delta_1\psi_\alpha(x) \\ &= ... + \sqrt{\theta}_{1\alpha}\delta_2 f - \sqrt{2}\theta_{2\alpha}\delta_1 f \\ = ... + 2i\theta_{1\alpha}\bar{\theta}_{2\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu\psi_\beta(x) &- 2i\theta_{2\alpha}\bar{\theta}_{1\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu\psi_\beta(x)\end{aligned} \tag{3.22}$$

Using Fierz identity $\chi_\alpha\bar{\eta}_{\dot{\alpha}} = \frac{1}{2}(\chi\sigma_\mu\bar{\eta})\sigma^\mu_{\alpha\dot{\alpha}}$ to rewrite $\theta_{1\alpha}\bar{\theta}_{2\dot{\alpha}}$ and $\theta_{2\alpha}\bar{\theta}_{1\dot{\alpha}}$, we will have

$$[\delta_1, \delta_2]\psi_\alpha(x) = ... + 0 \tag{3.23}$$

We say that now the SUSY algebra is realized *off-shell*. Next let us determine

$$\begin{aligned}[\delta_1, \delta_2]f(x) &= \delta_1\delta_2 f(x) - \delta_2\delta_1 f(x) \\ = i\sqrt{2}\bar{\theta}_{1\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\delta_2\psi_\alpha &- i\sqrt{2}\bar{\theta}_{2\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\delta_1\psi_\alpha(x) \\ = -2\bar{\theta}_{1\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^\nu_{\alpha\dot{\beta}}\bar{\theta}_2^{\dot{\beta}}\partial_\mu\partial_\nu\phi(x) &+ 2i\bar{\theta}_{2\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^\nu_{\alpha\dot{\beta}}\bar{\theta}_1^{\dot{\beta}}\partial_\mu\partial_\nu\phi(x) \\ +2i\bar{\theta}_{1\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\theta_{2\alpha}\partial_\mu f(x) &- 2i\bar{\theta}_{2\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\theta_{1\alpha}\partial_\mu f(x) \\ &= 2i((\bar{\theta}_1\bar{\sigma}^\mu\theta_2) - (\bar{\theta}_2\bar{\sigma}^\mu\theta_1)\partial_\mu f(x) \\ &\equiv 2i\left((\theta_1\sigma^\mu\bar{\theta}_2) - (\theta_2\sigma^\mu\bar{\theta}_2)\right)\partial_\mu f(x)\end{aligned} \tag{3.24}$$

after we have used the identity $\bar{\chi}\bar{\sigma}^\mu\eta = -\eta\sigma^\mu\bar{\chi}$.

To summarize, we will denote that the off-shell SUSY variation on fields off-shell are

$$\delta\phi(x) = \sqrt{2}\theta^\alpha\psi_\alpha(x) \tag{3.25}$$

$$\delta\psi_\alpha(x) = i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\phi(x) + \sqrt{2}\theta_\alpha f(x) \tag{3.26}$$

$$\delta f(x) = i\sqrt{2}\bar{\theta}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha(x) \tag{3.27}$$

3.2 SUSY invariant Lagrangian

The massless SUSY invariant Lagrangian within chiral multiplet (ϕ, ψ, f) sholud appear in the form

$$\mathcal{L}(\phi, \psi, f) = \partial_\mu\phi^*\partial^\mu\phi + i\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha + f^* f \tag{3.28}$$

$$S[\phi, \psi, f] = \int d^4x\mathcal{L}(\phi, \psi, f) \to \delta S[\phi, \psi, f] = 0 \tag{3.29}$$

but $\delta\mathcal{L} = \partial_\mu F(\phi, \psi, f) \neq 0$, and $\int d^4x \partial_\mu F = 0$ because of the boundary term. Let us determine

$$\begin{aligned}
\delta\mathcal{L} &= \partial_\mu\phi^*\partial^\mu\delta\phi + \partial_\mu\delta\phi^*\partial^\mu\phi \\
&\quad + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\delta\psi + i\delta\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + f^*\delta f + \delta f^* f \\
&= \sqrt{2}\partial_\mu\phi^*(\theta\partial^\mu\psi) + \sqrt{2}(\partial_\mu\bar{\psi}\bar{\theta})\partial^\mu\phi \\
&\quad - \sqrt{2}(\bar{\psi}\bar{\sigma}^\mu)(\sigma^\nu\bar{\theta})\partial_\mu\partial_\nu\phi + \sqrt{2}(\theta\sigma^\nu\bar{\sigma}^\mu\partial_\mu\psi)\partial_\nu\phi^* \\
&\quad + i\sqrt{2}(\bar{\psi}\bar{\sigma}^\mu\theta)\partial_\mu f + i\sqrt{2}f^*(\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi) \\
&\quad + i\sqrt{2}f^*(\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi) - i\sqrt{2}(\partial_\mu\bar{\psi}\sigma^\mu\theta)f \\
&= 0 + \sqrt{2}\partial_\mu(\theta\sigma^\nu\bar{\sigma}^\mu\psi)\partial_\nu\phi + 0 \\
&\quad + i\sqrt{2}\partial_\mu(f^*(\bar{\theta}\bar{\sigma}^\mu\psi)) - i\sqrt{2}\partial_\mu((\psi\sigma\bar{\theta})f) + 0
\end{aligned} \tag{3.30}$$

Then $\delta\mathcal{L}$ appears as the total derivative, so the action is SUSY invariant.