4 Superspace and Superfields

4.1 Superspace

QFT spacetime coordinates is denoted as $x^\mu = (t, \vec{x})$, its SUSY extension can be formulated on *superspace*, introduced by Salam and Strathdee (1978), with coordinates

$$(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \tag{4.1}$$

where $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ are Grassmannian coordinates with spinor indices $\alpha, \dot{\alpha}$. There basic properties are

$$\theta^2 = \theta^\alpha\theta_\alpha = \epsilon^{\alpha\beta}\theta_\beta\theta_\alpha = 2\theta_2\theta_1 = -2\theta_1\theta_2 \tag{4.2}$$

$$\bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\alpha}} = -2\bar{\theta}_{\dot{2}}\bar{\theta}_{\dot{1}} = 2\bar{\theta}_{\dot{1}}\bar{\theta}_{\dot{2}} \tag{4.3}$$

$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta^2,\ \ \theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2 \tag{4.4}$$

$$\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^2,\ \ \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}^2 \tag{4.5}$$

$$\theta_\alpha\bar{\theta}_{\dot{\alpha}} = \frac{1}{2}\underbrace{(\theta^\beta\sigma_{\mu\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}})}_{\theta\sigma_\mu\bar{\theta}}\sigma^\mu_{\alpha\dot{\alpha}} \tag{4.6}$$

$$\partial_\alpha\theta^\beta \equiv \frac{\partial\theta^\beta}{\partial\theta^\alpha} = \delta^\beta_\alpha,\quad \partial^\alpha\theta_\beta = -\delta^\alpha_\beta \tag{4.7}$$

$$\bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{\partial\bar{\theta}^{\dot{\beta}}}{\partial\bar{\theta}^{\dot{\alpha}}} = \delta^{\dot{\beta}}_{\dot{\alpha}},\ \ \bar{\partial}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{4.8}$$

where $(\partial_\alpha)^\dagger = \bar{\partial}_{\dot{\alpha}}$.

Superfunction $Y(x, \theta, \bar{\theta})$ is defined to be analytic function on superspace. Its infinitesimal supertranslation on superspace means

$$\theta \to \theta + \epsilon,\ \ \bar{\theta} \to \bar{\theta} + \bar{\epsilon} \tag{4.9}$$

One can write

$$\begin{aligned} &Y(x, \theta+\epsilon, \bar{\theta}+\bar{\epsilon}) \\ &= e^{-i(\epsilon Q+\bar{\epsilon}\bar{Q})}Y(x,\theta,\bar{\theta})e^{i(\epsilon Q+\bar{\epsilon}\bar{Q})} \quad (4.10)\\ &= e^{-i(\epsilon Q+\bar{\epsilon}\bar{Q})}e^{-i(xp+\theta Q+\bar{\theta}\bar{Q})}Y(0,0,0)e^{i(xp+\theta Q+\bar{\theta}\bar{Q})}e^{i(\epsilon Q+\bar{\epsilon}\bar{Q})} \quad (4.11)\end{aligned}$$

Let us determine

$$\begin{aligned} e^{i(\epsilon Q+\bar{\epsilon}\bar{Q})}e^{i(xp+\theta Q+\bar{\theta}\bar{Q})} &= e^{i(xP+(q+\epsilon)Q+(\bar{Q}+\bar{\epsilon})\bar{Q})-\frac{1}{2}[\bar{\theta}\bar{Q},\epsilon Q]-\frac{1}{2}[\theta Q,\bar{\epsilon}\bar{Q}]} \\ &= e^{i(xP+\theta Q+\bar{\theta}\bar{Q})-(\epsilon\sigma^\mu\bar{\theta})P_\mu-(\theta\sigma^\mu\bar{\epsilon})P_\mu} \\ &= e^{i(x+i(\epsilon\sigma^\mu\bar{\theta})+i(\theta\sigma^\mu\bar{\epsilon}))P_\mu+i(\theta+\epsilon)Q+i(\bar{\theta}+\bar{\epsilon})\bar{Q}} \end{aligned} \tag{4.12}$$

This means that the supertranslations result to the spacetime transformation in the form

$$\delta\theta=\epsilon,\ \ \delta\bar{\theta}=\bar{\epsilon}\to\delta x^\mu=i(\theta\sigma^\mu\bar{\epsilon})+i(\epsilon\sigma^\mu\bar{\theta}) \tag{4.13}$$

From (4.11) we will have

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}Y(x,\theta,\bar{\theta}) &= (i\theta\sigma^\mu\bar{\epsilon}+i\epsilon\sigma^\mu\bar{\theta})\partial_\mu Y(x,\theta,\bar{\theta}) \\ &+i\epsilon^\alpha\partial_\alpha Y(x,\theta,\bar{\theta})+i\bar{\epsilon}^{\dot{\alpha}}\partial_{\dot{\alpha}}Y(x,\theta,\bar{\theta}) \end{aligned} \tag{4.14}$$

Similarly from (4.10), we can have

$$\begin{aligned} &Y(x,\theta+\epsilon,\bar{\theta}+\bar{\epsilon}) \\ =(1-i(\epsilon Q+\bar{\epsilon}\bar{Q})+...)&Y(x,\theta,\bar{\theta})(1+i(\epsilon Q+\bar{\epsilon}\bar{Q})+...)-Y(x,\theta,\bar{\theta}) \\ &=-i\epsilon[Q,Y]-i\bar{\epsilon}[\bar{Q},Y] \end{aligned} \tag{4.15}$$

Let us define

$$[Y,Q_\alpha]=Q_\alpha Y,\ \ [Y,\bar{Q}_{\dot{\alpha}}]=\bar{Q}_{\dot{\alpha}}Y \tag{4.16}$$

$$\to\delta_{\epsilon,\bar{\epsilon}}Y=(i\epsilon Q+\bar{\epsilon}\bar{Q})Y \tag{4.17}$$

Under comparison with (4.14), we observe that

$$Q_\alpha=-i\partial_\alpha-\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \tag{4.18}$$

$$\bar{Q}_{\dot{\alpha}}=+i\bar{\partial}_{\dot{\alpha}}+\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{4.19}$$

$$\to\{Q_\alpha,Q_\beta\}=\{\bar{Q}_{\dot{\alpha}},\bar{Q}_{\dot{\beta}}\}=0,\{Q_\alpha,\bar{Q}_{\dot{\beta}}\}=2\sigma^\mu_{\alpha\dot{\beta}}P_\mu \tag{4.20}$$

4.2 Chiral superfields

Let us define the chiral operators

$$D_\alpha=\partial_\alpha+i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \tag{4.21}$$

$$\bar{D}_{\dot{\alpha}}=\bar{\partial}_{\dot{\alpha}}+i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{4.22}$$

$$\to\{D_\alpha,\bar{D}_{\dot{\beta}}\}=2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu=2\sigma^\mu_{\alpha\dot{\beta}}P_\mu \tag{4.23}$$

The chiral superfield $\Phi(x,\theta,\bar{\theta})$ is defined to satisfy a condition

$$\bar{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0 \tag{4.24}$$

The anti-chiral superfield $\Psi(x,\theta,\bar{\theta})$ is defined to satisfy a condition

$$D_\alpha\Psi(x,\theta,\bar{\theta}) = 0 \tag{4.25}$$

Let us define the chiral and anti-chiral coordinates, respectively, as

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \to \bar{D}_{\dot{\alpha}}y^\mu = 0 \tag{4.26}$$

$$\bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta} \to D_\alpha\bar{y}^\mu = 0 \tag{4.27}$$

So that the general form of chiral superfield can be written as

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) - \frac{1}{2}\theta^2F(y) \tag{4.28}$$

From (4.26), with Taylor's expansion for $\theta \to 0$, we have

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) \\ &+\sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \theta\theta F(x)\end{aligned} \tag{4.29}$$

And the corresponding anti-chiral superfield is $\bar{\Phi} = (\Phi)^\dagger$. Let us determine

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(y,\theta) = (i\epsilon Q + i\bar{\epsilon}\bar{Q})\Phi(y,\theta) \tag{4.30}$$

$$Q_\alpha = -i\partial_\alpha,\ \bar{Q}_{\dot{\alpha}} = -i\bar{\partial}_{\dot{\alpha}} + 2\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_{y^\mu} \tag{4.31}$$

$$\begin{aligned}\to \delta_{\epsilon,\bar{\epsilon}}\Phi(y,\theta) &= (\epsilon^\alpha\partial_\alpha + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\partial_{y^\mu}\Phi(y,\theta) \\ = \sqrt{2}\epsilon\psi(y) - 2\epsilon\theta F(y) &+ 2i\theta\sigma^\mu\bar{\epsilon}\left(\partial_{y^\mu}\phi(y) + \sqrt{2}\theta\partial_{y^\mu}\psi(y)\right) \\ = \sqrt{2}\epsilon\psi(y) &+ \sqrt{2}\theta\left(-\sqrt{2}\epsilon F(y) + \sqrt{2}i\sigma^\mu\bar{\epsilon}\partial_{y^\mu}\phi(y)\right) \\ &-\theta\theta\left(-i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_{y^\mu}\psi(y)\right)\end{aligned} \tag{4.32}$$

For SUSY invariant superfield $\delta_{\epsilon,\bar{\epsilon}}\Phi(y,\theta) =$, then we have SUSY transformations on component fields in the form

$$\delta\phi = \sqrt{2}\epsilon\psi \tag{4.33}$$

$$\delta\psi_\alpha = \sqrt{2}i\sigma^\mu_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\partial_\mu\phi - \sqrt{2}\epsilon_\alpha F \tag{4.34}$$

$$\delta F = i\sqrt{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon} \tag{4.35}$$

The SUSY in variant Lagrangian of the chiral superfield components will be

$$\mathcal{L} = \partial_\mu \phi^* \partial \phi + \bar{\psi}_{\dot{\alpha}} i \bar{\sigma}^{\mu\dot{\alpha}\beta} \partial_\mu \psi_\beta + F^* F \tag{4.36}$$

It is known as *Wess-Zumino* model. It can be proved to SUSY invariant, up to the total derivative term, by using (4.33-35).

4.3 Superpotential

Note that (4.36) is the simplest SUSY model of a free massless fields, where F is auxillary field without kinetic term. The interaction term can be inserted in form of *Kahler superpotential*

$$K(\Phi^\dagger \Phi)$$

K is said to be *canonical* if it is a polynomial of chiral superfield.

4.4 Vector superfield