4 Superspace and Superfields

4.1 Superspace

QFT spacetime coordinates is denoted as $x^\mu = (t, \vec{x})$, its SUSY extension can be formulated on *superspace*, introduced by Salam and Strathdee (1978), with coordinates

$$(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \tag{4.1}$$

where $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ are Grassmannian coordinates with spinor indices $\alpha, \dot{\alpha}$. There basic properties are

$$\theta^2 = \theta^\alpha\theta_\alpha = \epsilon^{\alpha\beta}\theta_\beta\theta_\alpha = 2\theta_2\theta_1 = -2\theta_1\theta_2 \tag{4.2}$$

$$\bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\alpha}} = -2\bar{\theta}_{\dot{2}}\bar{\theta}_{\dot{1}} = 2\bar{\theta}_{\dot{1}}\bar{\theta}_{\dot{2}} \tag{4.3}$$

$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta^2, \;\; \theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2 \tag{4.4}$$

$$\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^2, \;\;\; \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}^2 \tag{4.5}$$

$$\theta_\alpha\bar{\theta}_{\dot{\alpha}} = \frac{1}{2}\underbrace{\left(\theta^\beta\sigma_{\mu\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right)}_{\theta\sigma_\mu\bar{\theta}}\sigma^\mu_{\alpha\dot{\alpha}} \tag{4.6}$$

$$\partial_\alpha\theta^\beta \equiv \frac{\partial\theta^\beta}{\partial\theta^\alpha} = \delta^\beta_\alpha, \;\;\; \partial^\alpha\theta_\beta = -\delta^\alpha_\beta \tag{4.7}$$

$$\bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{\partial\bar{\theta}^{\dot{\beta}}}{\partial\bar{\theta}^{\dot{\alpha}}} = \delta^{\dot{\beta}}_{\dot{\alpha}}, \;\; \bar{\partial}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{4.8}$$

where $(\partial_\alpha)^\dagger = \bar{\partial}_{\dot{\alpha}}$.

Superfunction $Y(x, \theta, \bar{\theta})$ is defined to be analytic function on superspace. Its infinitesimal supertranslation on superspace means

$$\theta \to \theta + \epsilon, \;\; \bar{\theta} \to \bar{\theta} + \bar{\epsilon} \tag{4.9}$$

One can write

$$Y(x, \theta + \epsilon, \bar{\theta} + \bar{\epsilon})$$

$$= e^{-i(\epsilon Q + \bar{\epsilon}\bar{Q})}Y(x, \theta, \bar{\theta})e^{i(\epsilon Q + \bar{\epsilon}\bar{Q})} \tag{4.10}$$

$$= e^{-i(\epsilon Q + \bar{\epsilon}\bar{Q})}e^{-i(xp + \theta Q + \bar{\theta}\bar{Q})}Y(0,0,0)e^{i(xp + \theta Q + \bar{\theta}\bar{Q})}e^{i(\epsilon Q + \bar{\epsilon}\bar{Q})} \tag{4.11}$$

Let us determine

$$e^{i(\epsilon Q+\bar{\epsilon}\bar{Q})}e^{i(xp+\theta Q+\bar{\theta}\bar{Q})} = e^{i(xP+(q+\epsilon)Q+(\bar{Q}+\bar{\epsilon})\bar{Q})-\frac{1}{2}[\bar{\theta}\bar{Q},\epsilon Q]-\frac{1}{2}[\theta Q,\bar{\epsilon}\bar{Q}]}$$
$$= e^{i(xP+\theta Q+\bar{\theta}\bar{Q})-(\epsilon\sigma^\mu\bar{\theta})P_\mu-(\theta\sigma^\mu\bar{\epsilon})P_\mu}$$
$$= e^{i(x+i(\epsilon\sigma^\mu\bar{\theta})+i(\theta\sigma^\mu\bar{\epsilon}))P_\mu+i(\theta+\epsilon)Q+i(\bar{\theta}+\bar{\epsilon})\bar{Q}} \quad (4.12)$$

This means that the supertranslations result to the spacetime transformation in the form

$$\delta\theta = \epsilon,\ \delta\bar{\theta} = \bar{\epsilon} \rightarrow \delta x^\mu = i(\theta\sigma^\mu\bar{\epsilon}) + i(\epsilon\sigma^\mu\bar{\theta}) \quad (4.13)$$

From (4.11) we will have

$$\delta_{\epsilon,\bar{\epsilon}}Y(x,\theta,\bar{\theta}) = (i\theta\sigma^\mu\bar{\epsilon} + i\epsilon\sigma^\mu\bar{\theta})\partial_\mu Y(x,\theta,\bar{\theta})$$
$$+i\epsilon^\alpha\partial_\alpha Y(x,\theta,\bar{\theta}) + i\bar{\epsilon}^{\dot{\alpha}}\partial_{\dot{\alpha}}Y(x,\theta,\bar{\theta}) \quad (4.14)$$

Similarly from (4.10), we can have

$$Y(x,\theta+\epsilon,\bar{\theta}+\bar{\epsilon})$$
$$= (1 - i(\epsilon Q + \bar{\epsilon}\bar{Q}) + ...)Y(x,\theta,\bar{\theta})(1 + i(\epsilon Q + \bar{\epsilon}\bar{Q}) + ...) - Y(x,\theta,\bar{\theta})$$
$$= -i\epsilon[Q,Y] - i\bar{\epsilon}[\bar{Q},Y] \quad (4.15)$$

Let us define

$$[Y,Q_\alpha] = Q_\alpha Y,\ [Y,\bar{Q}_{\dot{\alpha}}] = \bar{Q}_{\dot{\alpha}}Y \quad (4.16)$$
$$\rightarrow \delta_{\epsilon,\bar{\epsilon}}Y = (i\epsilon Q + \bar{\epsilon}\bar{Q})Y \quad (4.17)$$

Under comparison with (4.14), we observe that

$$Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \quad (4.18)$$
$$\bar{Q}_{\dot{\alpha}} = +i\bar{\partial}_{\dot{\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \quad (4.19)$$
$$\rightarrow \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu \quad (4.20)$$

4.2 Chiral superfields

Let us define the chiral operators

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \quad (4.21)$$
$$\bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \quad (4.22)$$
$$\rightarrow \{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu \quad (4.23)$$

The chiral superfield $\Phi(x,\theta,\bar{\theta})$ is defined to satisfy a condition

$$\bar{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0 \tag{4.24}$$

The anti-chiral superfield $\bar{\Phi}(x,\theta,\bar{\theta})$ is defined to satisfy a condition

$$D_{\alpha}\bar{\Phi}(x,\theta,\bar{\theta}) = 0 \tag{4.25}$$

Let us define the chiral and anti-chiral coordinates, respectively, as

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \rightarrow \bar{D}_{\dot{\alpha}}y^\mu = 0 \tag{4.26}$$
$$\bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta} \rightarrow D_{\alpha}\bar{y}^\mu = 0 \tag{4.27}$$

So that the general form of chiral superfield can be written as

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) - \frac{1}{2}\theta^2 F(y) \tag{4.28}$$

From (4.26), with Taylor's expansion for $\theta \rightarrow 0$, we have

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) \\ &+\sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \theta\theta F(x)\end{aligned} \tag{4.29}$$

Let us determine

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(y,\theta) = (i\epsilon Q + i\bar{\epsilon}\bar{Q})\Phi(y,\theta) \tag{4.30}$$
$$Q_\alpha = -i\partial_\alpha,\ \bar{Q}_{\dot{\alpha}} = -i\bar{\partial}_{\dot{\alpha}} + 2\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_{y^\mu} \tag{4.31}$$

$$\begin{aligned}\rightarrow \delta_{\epsilon,\bar{\epsilon}}\Phi(y,\theta) &= (\epsilon^\alpha\partial_\alpha + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\partial_{y^\mu}\Phi(y,\theta)) \\ = \sqrt{2}\epsilon\psi(y) - 2\epsilon\theta F(y) &+ 2i\theta\sigma^\mu\bar{\epsilon}\left(\partial_{y^\mu}\phi(y) + \sqrt{2}\theta\partial_{y^\mu}\psi(y)\right) \\ = \sqrt{2}\epsilon\psi(y) &+ \sqrt{2}\theta\left(-\sqrt{2}\epsilon F(y) + \sqrt{2}i\sigma^\mu\bar{\epsilon}\partial_{y^\mu}\phi(y)\right) \\ &-\theta\theta\left(-i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_{y^\mu}\psi(y)\right)\end{aligned} \tag{4.32}$$

For SUSY invariant superfield $\delta_{\epsilon,\bar{\epsilon}}\Phi(y,\theta)$, then we have SUSY transformations on component fields in the form

$$\delta_\epsilon\phi = \sqrt{2}\epsilon\psi \tag{4.33}$$
$$\delta_\epsilon\psi_\alpha = \sqrt{2}i\sigma^\mu_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\partial_\mu\phi - \sqrt{2}\epsilon_\alpha F \tag{4.34}$$
$$\delta_\epsilon F = i\sqrt{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon} \tag{4.35}$$

A similar expression for anti-chiral superfield $\bar{\Phi}(x,\theta,\bar{\theta})$ will be

$$D_\alpha\bar{\Phi}=0,\quad \bar{y}^\mu=x^\mu-i\theta\sigma^\mu\bar{\theta} \tag{4.36}$$

$$\bar{\Phi}(\bar{y})=\phi^*(\bar{y})+\sqrt{2}\bar{\theta}\bar{\psi}(\bar{y})-\bar{\theta}\bar{\theta}F^*(\bar{y}) \tag{4.37}$$

After Taylor's expansion, we have

$$\begin{aligned}\bar{\Phi}(x,\theta,\bar{\theta})&=\phi^*(x)-i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^*(x)-\frac{1}{4}\theta\theta\bar{\theta}\partial^2\phi^*(x)\\&+\sqrt{2}\bar{\theta}\bar{\psi}(x)+\frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\psi(x)-\bar{\theta}\bar{\theta}F^*(x)\end{aligned} \tag{4.38}$$

And the supersymmetric variation $\delta_{\epsilon,\bar{\epsilon}}\bar{\Phi}$ will be

$$\delta_{\bar{\epsilon}}\phi^*=\sqrt{2}\bar{\epsilon}\bar{\psi} \tag{4.39}$$

$$\delta_{\bar{\epsilon}}\bar{\psi}=-i\sqrt{2}\epsilon\sigma^\mu\partial_\mu\phi^*-\sqrt{2}\bar{\epsilon}F^* \tag{4.40}$$

$$\delta_{\bar{\epsilon}}F^*=-i\sqrt{2}\epsilon\sigma^\mu\partial_\mu\bar{\psi} \tag{4.41}$$

4.3 Chiral superfield Lagrangian

The kinetic term of chiral multiplet can be constructed from chiral superfield in the form

$$\mathcal{L}_{kin}=\int d^2\theta d^2\bar{\theta}\bar{\Phi}(x,\theta,\bar{\theta})\Phi(x,\theta,\bar{\theta})=\bar{\Phi}\Phi|_{\bar{\theta}\bar{\theta}\theta\theta} \tag{4.42}$$

Let us determine

$$\begin{aligned}\bar{\Phi}(x,\theta,\bar{\theta})\Phi(x,\theta,\bar{\theta})&=[\phi^*(x)+\sqrt{2}\bar{\theta}\bar{\psi}(x)-\bar{\theta}\bar{\theta}F^*(x)-i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^*(x)\\&-\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi^*(x)+\frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}][\phi(x)+\sqrt{2}\theta\psi(x)-\theta\theta F(x)\\&+i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x)-\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x)-\frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta}]\end{aligned} \tag{4.43}$$

$$\rightarrow\bar{\Phi}\Phi|_{\theta\theta\bar{\theta}\bar{\theta}}=\theta\theta\bar{\theta}\bar{\theta}\left(-\frac{1}{2}\phi^*\partial^2\phi+\frac{i}{2}(\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi-\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi)+F^*F\right) \tag{4.44}$$

So that

$$\begin{aligned}\mathcal{L}_{kin}=\partial_\mu\phi^*(x)\partial^\mu\phi(x)&+\frac{i}{2}(\bar{\psi}(x)\bar{\sigma}^\mu\partial_\mu\psi(x)-\partial_\mu\bar{\psi}(x)\bar{\sigma}^\mu\psi(x))\\&+F^*(x)F(x)+total\ derivative\end{aligned} \tag{4.45}$$

This is known as Wess-Zumino model. A more general kinetic term can be constructed from the *Kahler superpotential* $K(\bar{\Phi},\Phi)$, a polynomial of its argument, in the form

$$K(\bar{\Phi},\Phi)=\sum_{n,m=1}^{N,M}c_{nm}\bar{\Phi}^n\Phi^m \tag{4.46}$$

$$\mathcal{L}_{kin}=\int d^2\theta d^2\bar{\theta}K(\bar{\Phi},\Phi)=K(\bar{\Phi},\Phi)|_{\bar{\theta}\bar{\theta}\theta\theta} \tag{4.47}$$

Its simplest form $K(\bar{\Phi},\Phi)=\bar{\Phi}(x,\theta,\bar{\theta})\Phi(x,\theta,\bar{\theta})$ is called *canonical Kahler superpotential*. and results to Wess-Zumino model above.

The interaction term can be derived from the superpotentials $W(\Phi),\bar{W}(\bar{\Phi})$, in the form

$$\mathcal{L}_{int}=\int d^2\theta W(\Phi)+\int d^2\bar{\theta}\bar{W}(\bar{\Phi})=W(\Phi)|_{\theta\theta}+\bar{W}(\bar{\Phi})|_{\bar{\theta}\bar{\theta}} \tag{4.48}$$

Let us determine the Taylor's expansion of $W(\Phi)$ about the scalar field component as

$$\begin{aligned}W(\Phi)&=W(\phi+\sqrt{2}\theta\psi-\theta\theta F)\\&=W(\phi)+\sqrt{2}\frac{\partial W}{\partial\phi}\theta\psi-\theta\theta\left(\frac{\partial W}{\partial\phi}F+\frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\psi\psi\right)\end{aligned} \tag{4.49}$$

$$\rightarrow W(\Phi)|_{\theta\theta}=-\frac{\partial W}{\partial\phi}F-\frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\psi\psi \tag{4.50}$$

Note that we have use the fact that $\theta^\alpha\psi_\alpha\theta^\beta\psi_\beta=-\frac{1}{2}\theta^\alpha\theta_\alpha\psi^\beta\psi_\beta$. Therefore

$$\mathcal{L}_{int}=-\frac{\partial W}{\partial\phi}F-\frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\psi\psi+h.c.=-W_\phi F-\frac{1}{2}W_{\phi\phi}\psi\psi+h.c. \tag{4.51}$$

The total Lagrangian, with the canonical Kahler superpotential, becomes

$$\begin{aligned}\mathcal{L}&=\bar{\Phi}\Phi_{\bar{\theta}\bar{\theta}\theta\theta}+W(\Phi)|_{\theta\theta}+\bar{W}(\bar{\Phi})|_{\bar{\theta}\bar{\theta}}\\&=\partial_\mu\phi^*\partial^\mu\phi+\frac{i}{2}(\partial_\mu\psi\sigma^\mu\bar{\psi}-\psi\sigma^\mu\partial_\mu\bar{\psi})+F^*F\\&\quad-W_\phi F-\frac{1}{2}W_{\phi\phi}\psi\psi-\bar{W}_{\bar{\phi}}F^*-\frac{1}{2}\bar{W}_{\bar{\phi}\bar{\phi}}\bar{\psi}\bar{\psi}\end{aligned} \tag{4.52}$$

The EOM of the scalar field F,F^*, from Euler-Lagrange equation, becomes

$$F^*=W_\phi,\ \ F=\bar{W}_{\bar{\phi}} \tag{4.53}$$

Bach insertion into (4.48), we get

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \frac{i}{2}(\partial_\mu\psi\sigma^\mu\bar{\psi} - \psi\sigma^\mu\partial_\mu\bar{\psi})$$
$$-|W_\phi|^2 - \frac{1}{2}W_{\phi\phi}\psi\psi - \frac{1}{2}\bar{W}_{\bar{\phi}\bar{\phi}}\bar{\psi}\bar{\psi} \tag{4.54}$$

There appears a non-trivial scalar potential in the form

$$V(\phi,\phi^*) = |W_\phi|^2 \tag{4.55}$$

For example of interacting Wess-Zumino model, with the superpotential

$$W(\Phi) = \frac{1}{2}m\Phi^2 \rightarrow \mathcal{L}_{int} = -m^2\phi^*\phi - \frac{1}{2}m\psi\psi - \frac{1}{2}m\bar{\psi}\bar{\psi} \tag{4.56}$$

It is the mass term.

4.4 Vector superfield

The real vector superfield $V(x,\theta,\bar{\theta})$, with a condition $V^\dagger = V$, its possible component fields are

$$V(x,\theta,\bar{\theta}) = C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x)$$
$$+\frac{i}{2}\theta\theta B(x) - \frac{i}{2}\bar{\theta}\bar{\theta}B^*(x)$$
$$+i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right)$$
$$-i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right)$$
$$+\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) - \frac{1}{2}\partial^2C(x)\right) \tag{4.57}$$

Note that real bosonic fields (C, M, N, v_μ, D) gives 8 bosonic degrees fo freedom, while the fermionic fields (χ, λ) give 8 fermionic degrees of freedom.

Let us apply with the *supergauge transformation* of the form

$$V \rightarrow V + i(\Phi^* + \Phi), \tag{4.58}$$

where $\Phi = (\phi, \psi, F)$ is chiral superfield, we will observe the transformations

$$C \to C + i(\phi^* + \phi) \tag{4.59}$$

$$\chi \to \chi - i\sqrt{2}\psi \tag{4.60}$$

$$B \to B - iF \tag{4.61}$$

$$v_\mu \to v_\mu + \partial_\mu(\phi + \phi^*) \tag{4.62}$$

$$\lambda \to \lambda \tag{4.63}$$

$$D \to D \tag{4.64}$$

In order to get gauge invariant vector field model, we have to choose ($C = 0, \chi = 0, B = 0$) components, this is called *Wess-Zumino gauge condition*. From (4.57) we will have

$$V(x, \theta, \bar{\theta}) = \theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{4.65}$$

or simply as $V = (v_\mu, \lambda, \bar{\lambda}, D)$. SUSY variations of these component fields are

$$\sqrt{2}\delta_\epsilon v^\mu = \epsilon\sigma^\mu\bar{\lambda} - \bar{\epsilon}\bar{\sigma}^\mu\chi \tag{4.66}$$

$$\sqrt{2}\delta_\epsilon\lambda = \epsilon D + \frac{i}{2}\sigma^\mu\bar{\sigma}^\nu(\partial_\mu v_\nu - \partial_\nu v_\mu) \tag{4.67}$$

$$\sqrt{2}\delta_\epsilon D = \epsilon\sigma^\mu\partial_\mu\lambda + \bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\lambda \tag{4.68}$$

4.5 Supersymmetric vector (gauge) field Lagrangian

In order to construct SUSY invariant vector field Lagrangian, we have to construct the super-field strength tensor of the form

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V, \quad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V \tag{4.69}$$

We can observe that they are invariant under super-gauge transformation

$$W_\alpha \to W_\alpha - \frac{i}{4}\bar{D}\bar{D}D_\alpha(\bar{\Phi} + \Phi) = 0 \tag{4.70}$$

$$\bar{W}_{\dot{\alpha}} \to \bar{W}_{\dot{\alpha}} - \frac{i}{4}DD\bar{D}_{\dot{\alpha}}(\bar{\Phi} + \Phi) = 0 \tag{4.71}$$

From (4.65) and (4.69) we will have

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \tag{4.72}$$

with $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. Now determine

$$\int d^2\theta W^\alpha W_\alpha = W^\alpha W_\alpha|_{\theta\theta} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + D^2 + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{4.73}$$

One thus finally have

$$\begin{aligned}\mathcal{L}_{gauge} &= \frac{1}{4}W^\alpha W_\alpha|_{\theta\theta} + \frac{1}{4}\bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}} \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}D^2\end{aligned} \tag{4.74}$$

This an *abelian supersymmetric gauge field Lagrangian.*

For the case of *non-abelian supersymmetric gauge field*, we will have

$$V = V^a T^a, \quad a = 1, 1, ..., dim(G) \tag{4.75}$$

where $\{T^a\}$ is a set of generators of gauge group G of dimension $dim(G)$. The super-gauge transformation will be written in the form

$$e^V \to e^{i\Phi^*} e^V e^{-i\Phi} \tag{4.76}$$

Within the Wess-Zumino gauge fixing condition the vector superfield, we will have the fact that

$$e^V = 1 + V + \frac{1}{2}V^2 \tag{4.77}$$

The super-field strength tensor are now written in the form

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\left(e^{-V}D_\alpha e^V\right), \; \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\left(e^V\bar{D}_{\dot{\alpha}}e^{-V}\right) \tag{4.78}$$

Under the super-gauge transformation

$$\begin{aligned}W_\alpha &\to -\frac{1}{4}\bar{D}\bar{D}\left(e^{i\Phi}e^{-V}e^{-i\Phi^*}D_\alpha e^{i\Phi^*}e^V e^{-i\Phi}\right) \\ &= -\frac{1}{4}e^{i\Phi}\bar{D}\bar{D}\left(e^{-V}D_\alpha e^V\right)e^{-i\Phi} = e^{i\Phi}W_\alpha e^{-i\Phi}\end{aligned} \tag{4.79}$$

Similarly

$$\bar{W}_{\dot{\alpha}} \to e^{i\Phi^*}\bar{W}_{\dot{\alpha}}e^{-i\Phi^*} \tag{4.80}$$

Now from (4.77), let us determine

$$
\begin{aligned}
W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\left[\left(1 - V + \frac{1}{2}V^2\right) D_\alpha \left(1 + V + \frac{1}{2}V^2\right)\right] \\
= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V - \frac{1}{8}\bar{D}\bar{D}D_\alpha V^2 + \frac{1}{4}\bar{D}\bar{D}V D_\alpha V \\
= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V - \frac{1}{8}\bar{D}\bar{D}V(D_\alpha V) - \frac{1}{8}\bar{D}\bar{D}(D_\alpha V)\cdot V + \frac{1}{4}\bar{D}\bar{D}V D_\alpha V \\
= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V + \underbrace{\frac{1}{8}\bar{D}\bar{D}[V, D_\alpha V]}_{additional\ term}
\end{aligned}
\quad (4.81)
$$

Let us determine the additional term

$$
\frac{1}{8}\bar{D}\bar{D}[V, D_\alpha V] = \frac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha[v_\mu, v_\nu] - \frac{i}{2}\theta\theta\sigma^\mu_{\alpha\dot{\beta}}[v_\mu, \bar{\lambda}^{\dot{\beta}}] \quad (4.82)
$$

Then we have

$$
W_\alpha = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu\mathcal{D}_\mu\bar{\lambda})_\alpha \quad (4.83)
$$

$$
\text{with } F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}[v_\mu, v_\nu], \quad \mathcal{D}_\mu = \partial_\mu - \frac{i}{2}[v_\mu,\ ] \quad (4.84)
$$

Insertion of gauge coupling constant, let us modify $V \to 2gV$ which results to

$$
v_\mu \to 2gv_\mu,\ \ \lambda \to 2g\lambda,\ \ D \to 2gD
$$

and then

$$
F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu], \quad \mathcal{D}_\mu = \partial_\mu - ig[v_\mu,\ ]
$$

The non-abelian supersymmetric gauge field (super Yang-Mills) Lagrangian is then written in the form

$$
\mathcal{L}_{susy-YM} = \frac{1}{4}\int d^2\theta Tr\left[W^\alpha W_\alpha\right] = Tr\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\mathcal{D}_\mu\bar{\lambda} + \frac{1}{2}D^2\right] \quad (4.85)
$$