14 Unitarity and Cutkosky Cuts

14.1 Unitary property of the S-matrix

The S-matrix is defined to be unitary operator satisfy unitary property

$$S^\dagger S = 1 \tag{14.1}$$

$$S = 1 + iM \mapsto M^\dagger M = i(M^\dagger - M) = 2Im[M] \tag{14.2}$$

Let us determine the Feynman propagator of scalar field

$$\begin{aligned}\Delta(p) &= \frac{1}{p^2 - m^2 + i\epsilon} \\ \mapsto Im\left[\Delta(p)\right] &= \frac{1}{2i}\left(\frac{1}{p^2 - m^2 + i\epsilon} - \frac{1}{p^2 - m^2 - i\epsilon}\right) \\ &= \frac{-\epsilon}{(p^2 - m^2)^2 + \epsilon^2}\end{aligned} \tag{14.3}$$

We observe that $Im[\Delta(p)] \xrightarrow{\epsilon \to 0} 0$, except around $p^2 = m^2$. It is said to be *on mass shell*. Let us determine its integrals over p^2

$$\int_0^\infty dp^2 \frac{-\epsilon}{(p^2 - m^2)^2 + \epsilon^2} = -\pi \tag{14.4}$$

So that one can write

$$Im[\Delta(p)] = -\pi\delta(p^2 - m^2)\theta(p^0) \tag{14.5}$$

14.2 Cutkosky cutting rules

When connect (14.5) to (14.2), we can say that if $M^\dagger M$ is the loop amplitude we can apply the cut to the loop propagator which becomes the imaginary part of the loop propagator that going on mass shell. This was known as *Cutkosky cut*. The cutting rules are defined as in the following

- cut through loop diagram in any way that can put the loop propagator on-shell
- for each cut we replace $\frac{1}{p^2 - m^2 + i\epsilon} \to -2\pi i\delta(p^2 - m^2)\theta(p^0)$
- sum over all cuts

$$2Im[M] = \sum_{cuts} M_{cut}$$

- the final result will be written in term of the *discontinuity* $Disc[T]$ of the diagram as

$$Disc[M] = 2Im[M] \tag{14.6}$$

This relation is hold order by order in perturbation theory.

Let us denote $M_n^{(L)}$ for L-loop and n-legs diagram, then we will have

$$\begin{aligned} Disc[M_4^{(0)}] &= 0 & (14.7) \\ Disc[M_4^{(1)}] &= M_4^{(0)\dagger} M_4^{(0)} & (14.8) \\ Disc[M_4^{(2)}] &= M_4^{(0)\dagger} M_4^{(1)} + M_4^{(1)\dagger} M_4^{(0)} + M_5^{(0)\dagger} T_5^{(0)} & (14.9) \end{aligned}$$

See the figure below.

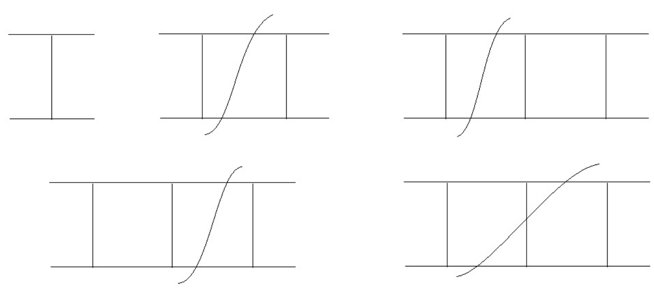

Figure 14.1: Cutkosky rules.

14.3 Dispersion relation

From Cauchy's integral formula,

$$f(\omega) = \frac{1}{2\pi i} \oint d\omega' \frac{f(\omega')}{\omega' - \omega} \tag{14.10}$$

when apply to the amplitude M which is a function of center of mass energy squared s, with a branch cut along the mass shell on the real line, we will have

$$M(s) = \frac{1}{2\pi i} \int_{s_0}^{\infty} ds' \frac{M(s+i\epsilon) - M(s-i\epsilon)}{s' - s} + \frac{1}{2\pi i} \int_{C_R + c_\epsilon} ds' \frac{M(s')}{s' - s} \tag{14.11}$$

where $s_0 = m^2$, see figure (14.2) below. The contributions from C_R and c_ϵ are zero when $|s| \to \infty$ and $\epsilon \to 0$, then we can have

$$M(s) = \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{Im[M(s')]}{s' - s} \tag{14.12}$$

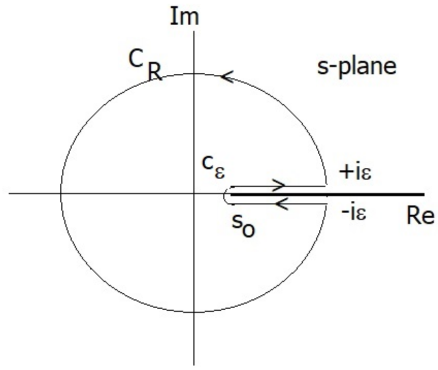

Figure 14.2: Dispersion relation.

14.4 One-loop amplitude with Cutkosky cut

14.5 One-loop scalar fields interaction

With the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \frac{1}{2}\partial_\mu\pi\partial^\mu\phi - g\phi\phi^2 \tag{14.13}$$

Then we have, fro massless scalar field ϕ,

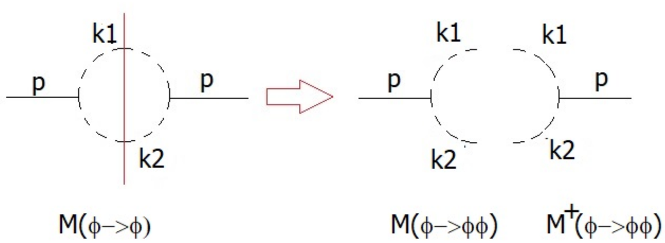

Figure 14.3: Cutkosky cut of one-loop scalar fields interaction.

$$2ImM(\phi \to \phi) = \int \frac{d^3k_1}{(2\pi)^3 2E_1}\int \frac{d^3k_2}{(2\pi)^3 2E_2}$$
$$\times(2\pi)^4\delta^{(4)}(k_1+k_2-p)|M(\phi\to\pi\pi)|^2 \tag{14.14}$$
$$= \int \frac{d^3k_1}{16\pi^2 E_1E_2}\delta(E_1+E_2-p^0)|M(\phi\to\pi\pi)|^2 \tag{14.15}$$

Since

$$M(\phi\to\pi\pi) = -ig$$

Then we have, in the CM-frame,

$$\mapsto 2Im[M(\phi \to \phi)] = \frac{g^2}{4} \int \frac{d|k|}{|k|} \delta(2|k| - M) = \frac{g^2}{4M} \tag{14.16}$$

$$(14.2) M(\phi \to \phi, s) = \frac{g^2}{4\pi M} \int_{s_0}^{\infty} ds' \frac{1}{s' - s} = \frac{g^2}{4\pi M} \frac{1}{s - s_0} \tag{14.17}$$

14.6 Generalized unitary