16 Quantization of Gauge Fields

Gauge fields can be quantized within functional integral form, since their gauge fixing conditions can be easily added into the formulation. The quantization is just writing the reduction of the generating functional in quadratic form.

16.1 Abelian gauge field

Let $A_\mu(x)$ be an abelian gauge field, with field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, its action functional is

$$S[A^\mu] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{16.1}$$

$$= \int d^4x \left(-\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu)\right)$$

$$= \int d^4x \left(\frac{1}{2}A^\mu(g_{\mu\nu}\partial_x^2 - \partial_\mu\partial_\nu)A^\nu\right) \tag{16.2}$$

where we have applied partial integration on the first term on the right hand side. This action is invariant under a *gauge transformation*

$$A^\mu(x) \to A^\mu_\alpha(x) = A^\mu(x) + \frac{1}{g}\partial^\mu\alpha(x) \tag{16.3}$$

$$\partial_\mu A^\mu = 0 \mapsto \partial_x^2\alpha = g(\partial_\mu A^\mu_\alpha) \tag{16.4}$$

where $\alpha(x)$ is a real scalar *gauge function*, and g is the gauge coupling constant, for the gauge fixing condition $\partial_\mu A^\mu = 0$. This shows that the gauge condition of $A^\mu_\alpha(x)$, i.e., $F[A^\mu_\alpha] = 0$ is derived as part of solution of equation of the gauge function (16.4).

The generating functional of gauge field is written in the form

$$Z[J_\mu] = \int \mathcal{D}[A^\mu]e^{iS[A^\mu]+i\int d^4x J_\mu(x)A^\mu(x)} \tag{16.5}$$

According to the identity of the delta function

$$\delta(f(x)) = \delta(x-a)\left|\frac{df}{dx}\right|^{-1}_{x=a}$$

The we have

$$\delta(F[A^\mu_\alpha]) = \delta(\alpha-\alpha_0)\det\left(\frac{\delta F[A^\mu_\alpha]}{\delta\alpha}\right)^{-1}_{\alpha_0} \equiv \delta(\alpha-\alpha_0)\Delta^{-1}_{FP}[A^\mu_\alpha] \qquad (16.6)$$

$$\mapsto 1 = \Delta_{FP}[A^\mu_{\alpha_0}]\int \mathcal{D}[\alpha]\delta(F[A^\mu_\alpha]) \qquad (16.7)$$

when α_0 is solution of equation of the gauge function at the required gauge fixing condition. Note that $\Delta_{FP}[A^\mu_\alpha]$ is known in the name of *Faddeev-Popov* determinant, and it is proved to be gauge invariant, i.e.

$$\Delta^{-1}_{FP}[A^\mu_{\alpha_0+\alpha'}] = \int \mathcal{D}[\alpha+\alpha']\delta(F[A^\mu_{\alpha+\alpha'}]) \equiv \Delta^{-1}_{FP}[A^\mu_{\alpha_0}] \qquad (16.8)$$

Insertion (16.7) into (16.5), after we set $\alpha_0 = 1$ we will get

$$Z[J_\mu] = \int \mathcal{D}[A^\mu]\left(\Delta_{FP}[A^\mu]\int \mathcal{D}[\alpha]\delta(F[A^\mu_\alpha]\right) e^{iS[A^\mu]+i\int d^4x J_\mu(x)A^\mu(x)} \qquad (16.9)$$

$$\equiv \int \mathcal{D}[\alpha]\int \mathcal{D}[A^\mu_\alpha]\Delta_{FP}[A^\mu_\alpha]\delta(F[A^\mu_\alpha])e^{iS[A^\mu_\alpha]+i\int d^4x J_\mu(x)A^\alpha_\mu(x)}$$

$$\xrightarrow{GT}\left(\int \mathcal{D}[\alpha]\right)\int \mathcal{D}[A^\mu]\Delta_{FP}[A^\mu]\delta(F[A^\mu])e^{iSA[\mu]+i\int d^4x J_\mu(x)A^\mu(x)} \qquad (16.10)$$

after we have done the gauge transformation $\alpha \to \alpha+\alpha'$, with invariant measure of the gauge function $\mathcal{D}[\alpha+\alpha'] = \mathcal{D}[\alpha]$, then we have set $\alpha+\alpha' = 1$. It appears with the stand alone functional integral measure of the gauge function which will be diverge to infinity. Anyway this divergence will not effect to our quantum field calculation with this generating functional and we can ignore it.

Now let us calculate $\Delta_{FP}[A^\mu]$ from the Lorentz gauge condition $\partial_\mu A^\mu(x) = 0$. From (16.4), we will have

$$F[A^\mu] = \partial_\mu A^\mu = \frac{1}{g}\partial^2_x\alpha(x) \mapsto \frac{\delta F[A^\mu(x)]}{\delta\alpha(y)} = \frac{1}{g}\delta^{(4)}(x-y)\partial^2_x \qquad (16.11)$$

$$\mapsto \Delta_{FP}[A^\mu] = \det\left(\frac{1}{g}\delta^{(4)}(x-y)\partial^2_x\right) \qquad (16.12)$$

And we can turn this into functional form as

$$\Delta_{FP}[A^\mu] = -i\int \mathcal{D}[c,\bar{c}]e^{\frac{i}{g}\int d^4x\int d^4y\delta^{(4)}(x-y)\bar{c}(y)\partial^2_x c(x)} \qquad (16.13)$$

$$= -i\int \mathcal{D}[c,\bar{c}]e^{\frac{i}{g}\int d^4x\bar{c}(x)\partial^2_x c(x)} \qquad (16.14)$$

where $c(x)$ is known as *Faddeev-Popov fermionic ghost field*, and $\bar{c}(x)$ is its conjugation. Then we have from (16.10)

$$\begin{aligned} Z[J_\mu] \;\; &\propto \;\; \int \mathcal{D}[c,\bar{c}] e^{\frac{i}{g}\int d^4x \bar{c}(x)\partial_x^2 c(x)} \\ & \qquad \times \int \mathcal{D}[A_\mu]\delta(F[A^\mu] e^{iS[A^\mu]+i\int d^4x J_\mu(x)A^\mu(x)} \end{aligned} \tag{16.15}$$

Again the fermionic functional integral appears as the stand alone part and independent to the gauge field functional integral. So that it can be ignored without any effect to the gauge field calculation.

Now let us turn the delta functional $\delta(F[A^\mu])$ into exponential form. We just shift the gauge fixing function with some scalar function $\omega(x)$ as

$$F[A^\mu(x)] \rightarrow F[A^\mu(x)] - \omega(x) \tag{16.16}$$

and then apply the functional integral average overall possible function $\omega(x)$ with Gaussian distribution in the form

$$\begin{aligned} Z_\xi[J_\mu] \;\; &\propto \;\; \int \mathcal{D}[A^\mu] e^{iS[A^\mu]+i\int d^4x J_\mu(x)A^\mu(x)} \\ & \qquad \times \int \mathcal{D}[\omega]\delta(F[A^\mu(x)] - \omega(x)] e^{-\frac{i}{2\xi}\int d^4x \omega^2(x)} \qquad (16.17) \\ & \propto \;\; \int \mathcal{D}[A^\mu] e^{iS[A^\mu]-\frac{i}{2\xi}\int d^4x F^2[A^\mu(x)]+i\int d^4x J_\mu(x)A^\mu(x)} \qquad (16.18) \end{aligned}$$

with ξ is a constant parameter. Now let us determine the modified gauge field action

$$\begin{aligned} & S[A^\mu] - \frac{1}{2\xi}\int d^4(\partial_\mu A^\mu(x))^2 \\ = \frac{1}{2}\int d^4x & \left(A^\mu(x)[g_{\mu\nu}\partial_x^2 - \partial_\mu\partial_\nu]A^\nu(x) - \frac{1}{\xi}\partial_\mu A^\mu(x)\partial_\nu A^\nu(x)\right) \\ & = \frac{1}{2}\int d^4x A^\mu(x)\left[g_{\mu\nu}\partial_x^2 - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]A^\nu(x) \qquad (16.19) \end{aligned}$$

The corresponding Green's function will be

$$\left[g_{\mu\nu}\partial_x^2 - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]G_\xi^{\mu\nu}(x,y) = \delta^{(4)}(x-y) \tag{16.20}$$

$$\xrightarrow{FT} -\left[k^2 g_{\mu\nu}\left(1-\frac{1}{\xi}\right)k_\mu k_\nu\right]G_\xi^{\mu\nu}(k) = 1 \tag{16.21}$$

$$\mapsto G_\xi^{\mu\nu}(k) = \frac{-1}{k^2+i\epsilon}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right] \tag{16.22}$$

Note that when setting $\xi = 1$ it is known as Feynman gauge, while $\xi = 0$ it is known as Landau gauge. From (16.17) we will have

$$Z_\xi[J_\mu] \propto \int \mathcal{D}[A^\mu] e^{\frac{i}{2}\int d^4x \int d^4y A^\mu(y) G^\xi_{\mu\nu}(x,y) A^\nu(x) + i\int d^4x J_\mu(x) A^\mu(x)} \tag{16.23}$$

$$\propto e^{\frac{i}{2}\int d^4x \int d^4y J_\mu(x) (G^{\mu\nu}_\xi(x,y))^{-1} J_\nu(y)} \tag{16.24}$$

This is the final form of the abelian gauge field generating functional.

16.2 Quantum Electrodynamics or QED

16.2.1 QED Lagrangian

QED action, with Feynman gauge and covariant derivative $D_\mu = \partial_\mu + ieA_\mu$, appears in the form

$$S[\psi, \bar{\psi}, A^\mu] = \int d^4x \left[\bar{\psi}(x)(i\not{D} - m)\psi(x) - \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \right] \tag{16.25}$$

where e is electric charge (coupling constant).

16.2.2 QED Feynman rules

16.2.3 Functional integral of QED

Propagators

Vertices

-TBA-

16.3 Non-abelian gauge field

Let $A_\mu(x)$ be a non-baelian gauge field, i.e.,

$$A_\mu = A^a_\mu t^a, \ \{t^a\} \in G, a = 1, 2, ..., \dim[G] \mapsto [t^a, t^b] = if^{abc} t^c \tag{16.26}$$

where G is the gauge group and $\{f^{abc}\}$ is a set of structure constants of G. Let us define the convariant derivative as

$$D_\mu = \partial_\mu - igA_\mu = \partial_\mu - igA_\mu^a t^a \tag{16.27}$$

$$\begin{aligned}
\mapsto [D_\mu, D_\nu] &= -ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right) \\
&= -ig\left(\partial_\mu A_\nu^a t^a - \partial_\nu A_\mu^a t^a - igA_\mu^b A_\nu^c [t^b, t^c]\right) \\
&= -ig\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c\right) t^a && (16.28) \\
&\equiv -igF_{\mu\nu}^a t^a = -igF_{\mu\nu} && (16.29) \\
\mapsto F_{\mu\nu} &= \frac{i}{g}[D_\mu, D_\nu] && (16.30)
\end{aligned}$$

The gauge field action is

$$\begin{aligned}
S[A_\mu] &= \int d^4x \left(-\frac{1}{2}Tr\left[F_{\mu\nu}F^{\mu\nu}\right]\right) \\
&= \int d^4x \left(-\frac{1}{2}F_{\mu\nu}^a F_b^{\mu\nu} \underbrace{Tr[t^a t^b]}_{=\frac{1}{2}\delta_b^a}\right) \\
&= \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu}\right) && (16.31)
\end{aligned}$$

This action is invariant under gauge transformation, with the gauge function $\beta(x) = \beta^a(x)t^a$, in the form

$$\begin{aligned}
A_\mu^a(x) \to A_\mu^{a\beta}(x) &= A_\mu^a(x) + \frac{1}{g}\partial_\mu\beta^a(x) + f^{abc}A_\mu^b(x)\beta^c(x) && (16.32) \\
&= A_\mu^a(x) + \frac{1}{g}D_\mu\beta^a(x) && (16.33) \\
\text{with } \partial_\mu A^{a\mu} = 0 \qquad &\mapsto \partial_\mu D^\mu \beta^a = g(\partial_\mu A_\mu^{a\beta}) && (16.34)
\end{aligned}$$

Similar to the abelian case,we can write its generating function in the form

$$\begin{aligned}
Z[J_\mu] &= \int \mathcal{D}[A_\mu]\left(\underbrace{\int \mathcal{D}[\beta]\Delta_{FP}[A_\mu]\delta(F[A_\mu^\beta])}_{=1}\right) e^{iS[A_\mu]+i\int d^4x J_\mu(x)A^\mu(x)} \\
&= \left(\int \mathcal{D}[\beta]\right)\int \mathcal{D}[A_\mu]\Delta_{FP}[A_\mu]\delta(F[A_\mu])e^{iS[A_\mu]+i\int d^4x J_\mu(x)A^\mu(x)}
\end{aligned} \tag{16.35}$$

when $F[A_\mu] = 0$ is the gauge fixing condition and the stand alone functional integral of gauge function $\left(\int \mathcal{D}[\beta]\right)$ will be ignored.

Similar to the abelian case, the Faddeev-Popov determinant can be written in fermionic functional integral from as

$$\begin{aligned} F[A_\mu] &= \partial_\mu A^\mu = \frac{1}{g}\partial_\mu D^\mu \beta \\ \mapsto \Delta_{FP}[A_\mu] = \det\left(\frac{\delta F[A_\mu(x)]}{\delta\beta(y)}\right) &= \frac{1}{g}\det\left(\delta^{4)}(x-y)\partial_\mu D^\mu\right) \qquad (16.36) \\ &= \frac{i}{g}\int \mathcal{D}[c,\bar{c}]e^{i\int d^4x\bar{c}(-\partial_\mu D^\mu)c(x)} \qquad (16.37) \end{aligned}$$

And from (16.35), we will have

$$Z[J_\mu] \propto \int \mathcal{D}[c,\bar{c}]\int \mathcal{D}[A_\mu]\delta(F[A_\mu])e^{iS[A_\mu]+i\int d^4x\bar{c}(x)(-\partial_\mu D^\mu)c(x)+i\int d^4xJ_\mu(x)A^\mu(x)} \qquad (16.38)$$

Note that in this case the fermionic Faddeev-Popov ghost field is coupling to gauge field through covariant derivative $D_\mu = \partial_\mu - igA_\mu$ inside its quadratic action kernel.

Next let us rewrite the delta function of the gauge fixing condition in exponential form by first shift the condition

$$F[A_\mu(x)] \to F[A_\mu(x)] - \omega(x) \qquad (16.39)$$

Then apply with Gaussian functional integral average over all possible $\omega(x)$ of (16.38) as

$$\begin{aligned} Z_\xi[J_\mu] \propto &\int \mathcal{D}[A_\mu,c,\bar{c}]\int \mathcal{D}[\omega]e^{-\frac{i}{2\xi}\int d^4x\omega^2(x)}\delta(F[A_\mu(x)]-\omega(x)) \\ &\times e^{iS[A_\mu]+i\int d^4x\bar{c}(x)(-\partial_\mu D^\mu)c(x)+i\int d^4xJ_\mu(x)A^\mu(x)} \qquad (16.40) \\ \propto &\int \mathcal{D}[A_\mu,c,\bar{c}]e^{iS[A_\mu]-\frac{i}{2\xi}\int d^4xF^2[A_\mu(x)]i\int d^4x\bar{c}(x)(-\partial_\mu D^\mu)c(x)} \\ &\times e^{i\int d^4xJ_\mu(x)A^\mu(x)} \qquad (16.41) \end{aligned}$$

Here we have gauge field action, with coupling to the ghost field, in the form

$$\begin{aligned} S[A_\mu,c,\bar{c}] &= S[A_\mu] - \frac{1}{2\xi}\int d^4x(\partial_\mu A^\mu(x))^2 \\ &= \int d^4x\left(\mathcal{L}_{YM}+\mathcal{L}_{GF}+\mathcal{L}_{ghost}\right) \qquad (16.42) \end{aligned}$$

where

$$\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu},\ \mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{16.43}$$

$$\mathcal{L}_{ghost} = \bar{c}(-\partial_\mu D^\mu)c = \bar{c}(-\partial_x^2)c + ig\bar{c}\partial_\mu A^\mu c \tag{16.44}$$

$$= \bar{c}^a(-\partial_x^2\delta^{ac})c^c + gf^{abc}\bar{c}^a\partial_\mu A^{b\mu}c^c$$

$$= \bar{c}^a(-\partial_x^2\delta^{ac})c^c - gf^{abc}(\partial_\mu\bar{c}^a)A^{b\mu}c^c \tag{16.45}$$

Here we do not need the final form of generating functional (16.41), but its final form of Lagrangian.

16.4 Quantum chromodynamics or QCD

QCD is *quantum chromodynamics*, it describes the interaction between color quarks with color gluon gauge fields within the color gauge group SU(3), i.e., gauge index $a = 1, 2, ..., 8 = dim(SU(3))$.

16.4.1 QCD Lagrangian

Its Lagrangian is

$$\mathcal{L}_{QCD} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2\xi}\partial_\mu A^{a\mu}\partial_\nu A^{a\mu}$$

$$+\bar{c}^a(-\partial^2)\delta^{ac}c^c - gf^{abc}(\partial_\mu\bar{c})A^{b\mu}c^c \tag{16.46}$$

$$= \bar{\psi}(i\not{\partial} - m)\psi + \frac{1}{2}A^{a\mu}\left[g_{\mu\nu}\partial_x^2 - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]A^{a\nu}$$

$$+g_c\bar{\psi}\gamma^\mu\psi A^a_\mu t^a - \frac{g}{2}f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu))A^{b\mu}A^{c\nu}$$

$$+\frac{g^2}{4}f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}$$

$$+\bar{c}^a(-\partial_x^2)c^a - gf^{abc}(\partial_\mu\bar{c}^a)A^{b\mu}c^c \tag{16.47}$$

16.4.2 QCD Feynman rules

16.4.3 Functional integral of QCD

Propagator

Vertices