1 Introduction

1.1 Natural unit

In this lecture we will work in natural unit of measurement, such that we will measure $c = 1 = \hbar$, this let us to measure everything in the mass unit as

$$[mass] = [energy] = [momentum] = [distance]^{-1} = [time]^{-1}$$

1.2 Mikowski space $\mathcal{M}^4$

To get Lorentz covariant formulation, we will place our formulation on 4-dimensional Monkowski space $\mathcal{M}^4$, with denoted spacetime position vector

$$x \in \mathcal{M}^4 \mapsto x = x^\mu e_\mu,\ x = x_\mu e^\mu,\ e_\mu e^\nu = \delta^\nu_\mu,\ \mu, \nu, ... = 0, 1, 2, 3$$

$$e_\mu e_\nu = g_{\mu\nu},\ e^\mu e^\nu = g^{\mu\nu},\ g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \mapsto g^{\mu\nu} = (g_{\mu\nu})^{-1}$$

$$g_{\mu\nu} := diag.(+, -, -, -),\ x^\mu := (x_0, \vec{x}) \mapsto x_\mu = g_{\mu\nu} x^\nu = (x_0, -\vec{x})$$

$$x^2 = g_{\mu\nu} x^\mu x^\nu = x_0^2 - \vec{x} \cdot \vec{x}$$

$$\partial_\mu = (\partial_0, \nabla), \partial^\mu = (\partial_0, -\nabla) \mapsto \partial^2 = \partial_\mu \partial^\mu = \partial_0^2 - \nabla^2$$

1.3 Special relativity and Lorentz transformation

Let us denote $x^\mu = (t, \vec{x})$ as a 4-position, its Lorentz transformation is

$$x^\mu \xrightarrow{LT} x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

In case of 01-Lorentz boost, we will have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \beta = \frac{u}{c}, \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

$$\mapsto t' = \gamma(t - \beta x),\ x' = \gamma(x - \beta t),\ y' = y,\ z' = z$$

$$\Delta t' = \gamma(\Delta t - \beta \Delta x) \xrightarrow{\Delta x = 0 \mapsto \Delta t' \Delta t = \Delta t_0} \gamma \Delta t_0$$

$$\Delta x' = \gamma(\Delta x - \beta \Delta t) \xrightarrow{\delta t = 0 \mapsto \Delta x \neq \Delta x_0} \Delta x' = \Delta x_0 = \gamma \Delta x$$

Relativistic kinematic

$$x^\mu = (t, \vec{x}) \mapsto x^2 = \tau^2$$

$$v^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = \gamma(1, \vec{v}) \mapsto v^2 = 1$$

$$p^\mu = m v^\mu = (\gamma m, \gamma m \vec{v}) = (E, \vec{p}) \mapsto p^2 = m^2$$

$$\mapsto E^2 = \vec{p}^2 - m^2,\ \vec{\beta} = \frac{\vec{p}}{E},\ \gamma = \frac{E}{m}$$

1.4 Lorentz symmetry, group, algebra, and representations

Let $f(x)$ be a Lorentz scalar function, i.e. invariant under Lorentz transformation as

$$x \xrightarrow{LT} x' = \Lambda x,\ f(x) \xrightarrow{LT} f'(x') = D(\Lambda) f(\Lambda x) = f(x)$$
$$\rightarrow f'(x) = f(\Lambda^{-1}x)$$

For infinitesimal LT we have $\Lambda^{\mu}{}_{\nu} \simeq \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}$ with $\omega^{\mu\nu} = -\omega^{\nu\mu}$, so that

$$f'(x) \simeq f(x - \omega x) = f(x) - \omega^{\mu\nu} x_{\nu}\partial_{\mu} f(x) + ...$$
$$\simeq f(x) - \frac{1}{2}\omega^{\mu\nu}(x_{\nu}\partial_{\mu} - x_{\mu}\partial_{\nu}) f(x) + ...$$
$$\simeq \left(1 - \frac{i}{2}\omega^{\mu\nu} M_{\mu\nu} + ...\right) f(x) = e^{-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}$$
$$\rightarrow M_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$$

where $M_{\mu\nu}$ is known as the generator of Lorentz transformation and it satisfies the algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = ig_{\mu\rho}M_{\nu\sigma} + ig_{\nu\sigma}M_{\mu\rho} - ig_{\mu\sigma}M_{\nu\rho} - ig_{\nu\rho}M_{\mu\sigma}$$

We can refine this algebra by introducing the new generators as

$$M_{0i} = K_i, M_{ij} = \frac{1}{2}\epsilon_{ijk}J_k$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k,\ [K_i, J_j] = i\epsilon_{ijk}K_k,\ [J_i, J_j] = i\epsilon_{ijk}J_k$$

We observe two coupled generalized angular momenta, we can decoupling them by doing the linear combination

$$J_i^{\pm} = \frac{1}{2}(J_i \pm iK_i) \mapsto [J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk}J_k^{\pm},\ [J_i^{\pm}, J_j^{\mp}] = 0$$
$$(J^+)^2|j_1, m_1\rangle = j_1(j_1+1)|j_1, m_1\rangle, (J^-)^2|j_2, m_2\rangle = j_2(j_2+1)|j_2, m_2\rangle$$

with $j+1, j+2 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...$ are generalized angular momentum quantum number. Note that (j_1, j_2) will be used for the Lorentz (matrix) representation, with total genralized angular momentum (spin) $j_1+j_2, j_1+j_2-1, ..., |j_1-j_2| \geq 0$. The following are the first few of its representations:

(j_1, j_2)	Spin	Name	Notation
$(0,0)$	1	Klein-Gordon scalar	φ
$(\frac{1}{2}, 0)$	$\frac{1}{2}$	L-Weyl spinor	ψ_{α}
$(0, \frac{1}{2})$	$\frac{1}{2}$	R-Weyl spinor	$\psi^{\dot{\alpha}}$
$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$	$\frac{1}{2}$	Dirac bi-spinor	Ψ
$(\frac{1}{2}, \frac{1}{2})$	1,(0)	Maxwell vector+gauge fixing	A^{μ}
$(\frac{1}{2}, \frac{1}{2}) \otimes \{(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\}$	$\frac{3}{2}, (\frac{1}{2})$	Rarita-Schwinger+condition	Ψ^{μ}
$(1,1)$	2, (0)	Einstein tensor+gauge fixing	$h^{\mu\nu}$

Note that another name of Rarita-Schwinger field is *gravitinos*, it is the superpartner of *graviton*, which is another name of Einstein tensor field.

1.5 Classical field dynamics

Let us denote $F(x)$ as a generic field function, i.e. a continuous function within spacetime volume of interest. Its dynamics is determined from the field Lagrangian which is written in term of Lagrangian density as

$$L = \int d^3x \mathcal{L}(F, \partial_\mu F) \mapsto S[F] = \int dt \int d^3x \mathcal{L} \tag{1.1}$$

From least action principle we get its equation of motion from Euler-Lagrange equation as

$$\begin{aligned} \delta S[F] = 0 = \int d^4x \delta\mathcal{L}(F, \partial_\mu F) &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial F}\delta F + \frac{\partial \mathcal{L}}{\partial \partial_\mu F}\delta \partial_\mu F \right) \\ &= \int d^4x \partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu F}\delta F \right)}_{=0 \ on \ Boundary} + \int d^4x \left(\frac{\partial \mathcal{L}}{\partial F} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu F} \right) \delta F \qquad (1.2) \\ &\to \frac{\partial \mathcal{L}}{\partial F} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu F} = 0 \text{ (inside spacetime volume)} \qquad (1.3) \end{aligned}$$

The conjugate momentum field is derived from the Lagrangian density as

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \partial_0 F} \tag{1.4}$$

The field Hamiltonian density is derived from Legendre transformation of the Lagrangian density as

$$\mathcal{H}(F, \nabla F, \pi) = \pi \partial_0 F - \mathcal{L} \mapsto H = \int d^3x \mathcal{H} \tag{1.5}$$

The following are some of classical fields appear in Lorentz representation:

a) Klein-Gordon scalar field: Let $\phi(x)$ be a free real scalar field, i.e. $\phi^* = \phi$, its satisfy Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0 \tag{1.6}$$

This equation is derived from the Lagrangian

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 \tag{1.7}$$

Its Hamiltonian is

$$\pi = \partial_0 \phi \mapsto H = \frac{1}{2}\int d^3x \left(\pi^2 + \nabla\phi \cdot \nabla\phi + m\phi^2 \right) \tag{1.8}$$

From (1.6), its trial free field solution is $\phi(x) \sim a(k)e^{-ik\cdot x}$, so that

$$(-k^2 + m^2)a(k) = 0 \mapsto 0 = -k^2 + m^2 = -\omega^2 + \vec{k}^2 + m^2, \tag{1.9}$$

$$(Dispersion) \ \to \omega = \omega_k = \sqrt{\vec{k}^2 + m^2} \tag{1.10}$$

The general free field solution will be

$$\phi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left(a(k)e^{-ik\cdot x} + a^*(k)e^{ik\cdot x}\right)(2\pi)\delta(\omega^2 - \omega_k^2) \tag{1.11}$$

Using identity

$$\delta(f(x)) = \sum_i \frac{\delta(x-a_i)}{|f'(a_i)|}, \text{ with } f(a_i) = 0$$

Then we have

$$\delta(\omega^2 - \omega_k^2) = \frac{1}{2\omega_k}(\delta(\omega - \omega_k) + \delta(\omega + \omega_k))$$

From (1.1.), we have

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k)e^{-ik\cdot x} + a^*(k)e^{ik\cdot x}\right)_{\omega=\omega_k} \tag{1.12}$$

In case of complex scalar field, i.e. $\phi^* \neq \phi$, its EOM is the same as (1.6) but its Lagrangian density is

$$\mathcal{L}_{KG} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \tag{1.13}$$

With $\pi = \partial_0\phi^*$ we have

$$H = \int d^3x \left(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi\right) \tag{1.14}$$

Its free field solution is

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k)e^{-ik\cdot x} + b^*(k)e^{ik\cdot x}\right)_{\omega=\omega_k} \tag{1.15}$$

b) Dirac spinor field: Let $\psi(x)$ be Dirac spinor satisfies Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{1.16}$$

where $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ is Dirac gamma matrix defined as

$$\gamma^0 = \beta = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \ \gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \ i = 1,2,3 \tag{1.17}$$

$$\text{Clifford algebra} \rightarrow \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{1.18}$$

With

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac Lagrangian is written as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{1.19}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ is called *Dirac conjugation.* The conjugate momentum is

$$\pi = i\bar{\psi}\gamma^0 = i\psi^\dagger \mapsto \mathcal{H} = -i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi \tag{1.20}$$

From (1.16), its trial free field solution for positive energy $E > 0$ is $\psi(x) \sim U(k,s)e^{-ik\cdot x}$, with spin quantum number s, then we have from (1.16)

$$0 = (\gamma^\mu k_\mu - m)U(k,s) = (\gamma^0 k^0 - \gamma^i k^i - m)U(k,s) \tag{1.21}$$

$$U(k,s) = \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} \mapsto 0 \begin{pmatrix} E-m & \vec{\sigma}\cdot\vec{k} \\ -\vec{\sigma}\cdot\vec{k} & E+m \end{pmatrix} \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} \tag{1.22}$$

$$(E-m)u_1 + \vec{\sigma}\cdot\vec{k}u_2 = 0, \; -\vec{\sigma}\cdot\vec{k}u_1 + (E+m)u_2 = 0$$

$$\to u_2 = \frac{\vec{\sigma}\cdot\vec{k}}{E+m}u_1, \; u_1(s) = \chi_s \; (spinor\ basis: \sum_s \chi_s^\dagger\chi_s = 1) \tag{1.23}$$

$$U(k,s) = N^+ \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \end{pmatrix} \tag{1.24}$$

where N is the normalization:

$$\sum_s U^\dagger U = |N^+|^2 \left(1 + \frac{(\vec{\sigma}\cdot\vec{k})^2}{(E+m)^2}\right) \sum_s \chi_s^\dagger\chi_s = 2E$$

with the identity $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{a}\times\vec{b}$, we get

$$\sum_s U^\dagger U = |N^+|^2(1 + \frac{\vec{k}\cdot\vec{k}}{E+m)^2}) = |N^+|^2\frac{2E}{E+m} = 2E \mapsto N^+ = \sqrt{E+m}$$

Exercise 1.1: Show that $\sum_s \bar{U}U = 2m$.

From (1.24), we can derive the completeness relation of the Dirac U-spinor in the form

$$U(k,s) = \begin{pmatrix} \sqrt{E+m}\chi_s \\ \frac{\vec{\sigma}\cdot\vec{k}}{\sqrt{E+m}}\chi_s \end{pmatrix}, \; \bar{U}(k,s) = \begin{pmatrix} \sqrt{E+m}\chi_s^\dagger & -\frac{\vec{\sigma}\cdot\vec{k}}{\sqrt{E+m}}\chi_s^\dagger \end{pmatrix} \tag{1.25}$$

$$\mapsto \sum_s U(k,s)\bar{U}(k,s) = \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & -\frac{(\vec{\sigma}\cdot\vec{k})^2}{E+m} \end{pmatrix} \sum_s \chi_s\chi_s^\dagger$$

$$= \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & E-m \end{pmatrix} \equiv \gamma^\mu k_\mu + m = \not{k} + m \tag{1.26}$$

On the other hand the trial free field negative energy $E < 0$ solution is

$\psi(x) \sim V(k,s)e^{ik\cdot x}$, then we have from (1.16)

$$0 = (-\gamma^\mu k_\mu - m)V(k,s) = -(\gamma^0 k^0 - \gamma^i k^i + m)V(k,s) \quad (1.27)$$

$$V(k,s) = \begin{pmatrix} v_1(s) \\ v_2(s) \end{pmatrix} \mapsto 0 = \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & -E+m \end{pmatrix} \begin{pmatrix} v_1(s) \\ v_2(s) \end{pmatrix} \quad (1.28)$$

$$\to (E+m)v_1(s) = -\vec{\sigma}\cdot\vec{k}v_2(s) = 0,\ \vec{\sigma}\cdot\vec{k}v_1(s) + (-E+m)v_2(s) = 0$$

$$v_1(s) = \frac{\vec{\sigma}\cdot\vec{k}}{E+m}v_2(s),\ v_2(s) = \chi_s,\ (spinor\ basis: \sum_s \chi_s^\dagger \chi_s = 1) \quad (1.29)$$

$$\to V(k,s) = N^- \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi_s \\ \chi_s \end{pmatrix} \quad (1.30)$$

Similarly, we can have

$$\sum_s V^\dagger(k,s)V(k,s) = 2E \mapsto N^- = \sqrt{E+m} \quad (1.31)$$

$$\sum_s V(k,s)\bar{V}(k,s) = \gamma^\mu k_\mu - m \equiv \not{k} - m \quad (1.32)$$

Exercise 1.2: Derive (1.31) and (1.32) explicitly.

The general free field solution of Dirac spinor is written, with Dirac dispersion $\omega = \omega_k = \pm\sqrt{\vec{k}^2 + m^2}$, in the form

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \sum_s \left\{a(k,s)U(k,s)e^{-ik\cdot x} + b^\dagger(k,s)V(k,s)e^{ik\cdot x}\right\} \times (2\pi)\delta(\omega^2 - \omega_k^2) \quad (1.33)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \sum_s \left\{a(k,s)U(k,s)e^{-ik\cdot x} + b^\dagger(k,s)V(k,s)e^{ik\cdot x}\right\}_{\omega=\omega_k} \quad (1.34)$$

Exercise 1.3: Discuss the existence of Dirac dispersion $\omega_k = \pm\sqrt{\vec{k}^2 + m^2}$.

1.6 Maxwell massless vector field

Let us denote $A^\mu = (\phi, \vec{A})$ as the massless Maxwell vector field, it Lagrangian is written in term of field strength tensor as

$$F^{\mu\nu} = \partial^\nu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^1 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix},\ F^{\mu\nu} = -F^{\nu\mu} \quad (1.35)$$

$$\mapsto \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu F^{\mu\nu} \to EoM: \partial_\mu F^{\mu\nu} = \partial^2 A^\nu = 0 \quad (1.36)$$

with the Lorentz gauge condition $\partial_\mu A^\mu = 0$. The conjugate field momentum is

$$\pi^\mu = \frac{\partial\mathcal{L}}{\partial\partial_0 A_\mu} = -F^{0\mu} \mapsto \pi^0 = 0,\ \pi^i = -\pi^{0i} = E^i \quad (1.37)$$

This means that there is no dynamics of temporal component of the vector field, so we can set $A^0 = 0$ for convenient.

The field Hamiltonian is

$$\mathcal{H} = -\frac{1}{2}\pi_i\pi^i + \frac{1}{2}F_{ij}F^{ij} = \frac{1}{2}E^iE^i + \frac{1}{2}\epsilon^{ijk}\epsilon^{ijl}B^kB^l$$
$$\mapsto H = \frac{1}{2}\int d^3x(\vec{E}^2 + \vec{B}^2) \tag{1.38}$$

From (1.36), the trial free field solution is $A^\mu(x) \sim \epsilon^\mu(k,\lambda)e^{-ik\cdot x}$, then we have

$$-k^2\epsilon^\mu(k,\lambda) = 0 \mapsto 0 = k^2 = \omega^2 - \vec{k}^2 \mapsto \omega^2 - \omega_k^2 = 0 \tag{1.39}$$

with Maxwell dispersion $\omega_k = |\vec{k}|$. The general free field solution is written in the form

$$A^\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^3}\int\frac{d\omega}{2\pi}\sum_\lambda \epsilon^\mu(k,\lambda)\left(a(k,\lambda)e^{-ik\cdot x} + a^*(k,\lambda)e^{ik\cdot x}\right)(2\pi)\delta(\omega^2 - \omega_k^2)$$
$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\sum_\lambda \epsilon^\mu(k,\lambda)\left(a(k,\lambda)e^{-ik\cdot x} + a^*(k,\lambda)e^{ik\cdot x}\right)_{\omega=\omega_k} \tag{1.40}$$

where $\epsilon^\mu(k,\lambda)$ is the polarization tensor, i.e. $\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon^\mu(k',\lambda) = \delta_{kk'}$.

1.7 Canonical quantization

From Hamiltonian dynamics of classical point particle, the dynamical variables are the degree of freedom $\{q\}$ and its conjugated momentum $\{p\}$. These are known as *canonical variables*, where as the particle Hamiltonian is continuous function of these variables, i.e., $H = H(q,p)$ and its dynamical equation is known as *Hamilton equations.*

$$\dot{q} = \frac{\partial H}{\partial p} = [q, H],\ \dot{p} = -\frac{\partial H}{\partial q} = [p, H]$$

where $[A,B] = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$ is called *Poisson bracket*, and $[q,p] = 1$ is satisfied for canonical variables.

Dynamics of particle can be determined from Hamiltonian curve on phase space, where set of canonical variables is its coordinate. For a closed system, i.e., constant energy, the Hamiltonian will be a closed curve on phase space.

Canonical quantization, the change from classical to quantum dynamics, can be done by promoting the canonical variables to be canonical operators and constrained with some algebra

$$q \to \hat{q},\ p \to \hat{p} \to [\hat{q},\hat{p}] = i\hbar$$

Normally these operators satisfy eigen-value equations

$$\hat{q}|q\rangle = q|q\rangle,\ \hat{p}|p\rangle = p|p\rangle$$

$$\hat{q}|p\rangle = i\partial_p|p\rangle,\ \hat{p}|q\rangle = -i\partial_q|q\rangle,\ \langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipq/\hbar}$$

The classical Hamiltonian also becomes Hamiltonian operator and has its own eigen-value equation

$$H \to \hat{H},\ \hat{H}|\psi_E\rangle = E|\psi_E\rangle$$

This is known in the name of Schrodinger's equation. Note that the change from Poisson bracket of canonical variables in classical particle to be commutation algebra of canonical variables of quantum particle is an importance process of canonical quantization. Let $A(q,p)$ be some physical property of classical particle, its quantum dynamics is determined from its operator picture satisfy Heisenberg's equation

$$\partial_t\hat{A} == \frac{1}{i\hbar}[\hat{A},\hat{H}]$$

1.8 Poincare symmetry, group, algebra and representations

Poincare transformation is known as inhomogeneous Lorentz transformation, i.e. Lorentz rotation plus translation. Its generator compose of P_μ for translation and $M_{\mu\nu}$ for Lorentz rotation, which satisfy the algebra

$$[P_\mu, P_\nu] = 0 \tag{1.41}$$

$$[P_\mu, M_{\rho\sigma}] = ig_{\mu\sigma}P_\rho - ig_{\mu\rho}P_\sigma \tag{1.42}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = ig_{\mu\rho}M_\sigma + ig_{\nu\sigma}M_{\mu\rho} - ig_{\mu\sigma}M_{\nu\rho} - ig_{\nu\rho}M_{\mu\sigma} \tag{1.43}$$

Two Casimir operators of these algebra are

$$C_1 = P^2,\ C_2 = W^2,\ W^\mu = \frac{1}{2}\epsilon^{\mu\rho\sigma\nu}M_{\rho\sigma}P_\nu \tag{1.44}$$

where W^μ is known as *Pauli-Lubanski vector*. There are two representations of Poincare algebra:

a)Massive representation: For massive particle, on its rest frame we will have $P^\mu = (M,0,0,0)$. So that

$$C_1 = M^2,\ W^\mu = \frac{1}{2}M\epsilon^{\mu\rho\sigma 0}M_{\rho\sigma} \mapsto W^i = \frac{1}{2}M\epsilon^{ijk}M_{jk} = MJ^i \tag{1.45}$$

$$C_2 = M^2J^2 \equiv M^2j(j+1),\ j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ... \equiv s(spin) \tag{1.46}$$

The eigen basis for massive representation will be identified with particle mass ans spin, i.e., $|M,s\rangle$.

b)Massless representation: For massless particle, on its light front frame in 3-direction we will have $P^\mu = (E,0,0,E)$. So that

$$C_1 = 0, W^\mu = \frac{1}{2}E\epsilon^{\mu\rho\sigma 0}M_{\rho\sigma} - \frac{1}{2}E\epsilon^{\mu\rho\sigma 3}M_{\rho\sigma} \tag{1.47}$$

$$\to W^0 = -\frac{1}{2}E\epsilon^{0ij}M_{ij},\ W^3 = \frac{1}{2}E\epsilon^{3ij}M_{ij},\ i,j = 1,2 \tag{1.48}$$

We observe that W^0 generate left- handedness rotation on 12-plane and W^3 generate right-handedness rotation on 12-plane. They are *helicity operators* $\hat{h}$ for massless particle propagate in 3-direction, with helicity number $h = \pm 1$. So that the eigen-basis for massless representation of Poincare algebra will be denoted with particle energy E ans its helicity h, i.e. $|E, h\rangle$.

Note that the eigen-basis of massive representation is derived from the eigen-vector of the (spin) angular momentum which are generators of SO(3) group, while the eigen-basis for massless representation is derived from the eigen-vector of helicity operator which generate rotation on plane perpendicular to particle propagation or SO(2) or Euclidean E group. These sugroups of Poincare group are known as *Wigner little groups*.